

Name:

ID:

MCQs (5 marks). Tick/circle the best answer.

- Suppose a polynomial p_n interpolates a function f at $n + 1$ distinct nodes. Which of the following statements is true?
 - There are infinitely many such polynomials
 - The interpolating polynomial always has degree exactly n
 - ✓ The interpolating polynomial is unique
 - The interpolation error is zero everywhere
- How many interpolation nodes are required to uniquely determine a polynomial of degree at most n ?
 - n
 - ✓ $n + 1$
 - $2n$
 - Infinitely many
- Let $f(x) = \sin x$. Which of the following is the degree-3 Taylor polynomial of f about $x_0 = 0$?
 - x
 - $x - \frac{x^3}{3}$
 - ✓ $x - \frac{x^3}{3!}$
 - $x - \frac{x^3}{6} + \frac{x^5}{120}$
- Which of the following best explains why Vandermonde interpolation is avoided in practice?
 - The interpolating polynomial does not exist
 - ✓ Solving the system is expensive
 - The Vandermonde matrix is not invertible
 - The method only works for linear polynomials
- The key computational advantage of Newton's interpolation form is that:
 - It avoids floating-point arithmetic
 - ✓ New nodes can be added with $O(n)$ work
 - It minimizes interpolation error
 - It requires no multiplications

Question 1: (4+5+1 marks) Suppose we have a function $f(x) = x \sin x$.

- What is the degree-2 Taylor polynomial p_2 near $x_0 = 0$? (Hint: $p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$.)
- Find the degree-2 interpolating polynomial p_2 , with the nodes $x_0, x_1, x_2 = 0, 1, -1$. You can use either Vandermonde method, or Lagrange method, or Newton's method.
- Which one of the polynomials in (a) or (b) gives the least absolute error while calculating $f(0.5)$? (Hint: absolute error = $|f(0.5) - p_2(0.5)|$, and you have to compare the quantities for both the polynomials in (a) and (b)).

Solution. (a) $f(0) = 0$; $f'(x) = x \cos x + \sin x$, so $f'(0) = 0$.

$$f''(x) = (x \cos x + \sin x)' = -x \sin x + \cos x + \cos x \implies f''(0) = 2.$$

$$\text{Therefore, } p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = x^2.$$

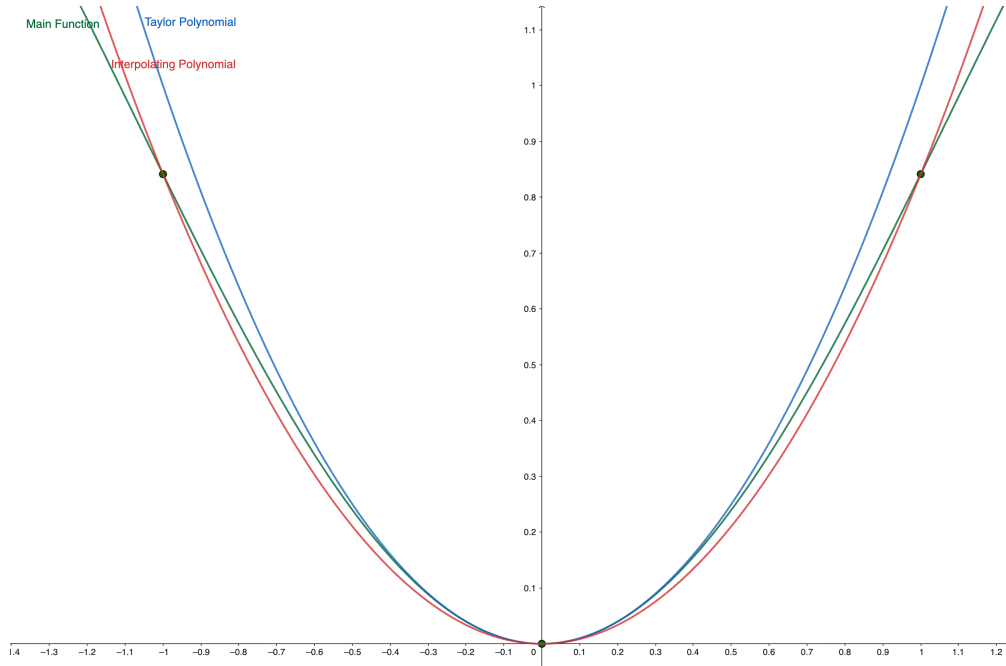
(b) The nodes are $(0, 0), (1, f(1)), (-1, f(-1))$. We shall use Vandermonde method:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix}^{-1} \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ f(1) \\ f(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sin 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.841 \end{bmatrix}$$

$$\text{Therefore, } p_2(x) = 0.841 x^2.$$

- The absolute error for (a) is $|0.5 \sin(0.5) - (0.5)^2| = 0.0103$. The absolute error for (b) is $|0.5 \sin(0.5) - 0.841(0.5)^2| = 0.0295$. Hence, (a) gives a better result.

For your understanding, here is a graphical representation of all three functions:



As you can see, near 0, the Taylor polynomial gives a much better estimate of the main function. However, near ± 1 , the interpolating polynomial has a smaller error.

Also, if you wanna solve (b) using Lagrange interpolation: you don't need to compute ℓ_0 , since $f(x_0) = 0$.

$$\ell_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{x(x + 1)}{2}, \quad \ell_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{x(x - 1)}{2}.$$

Therefore,

$$p_2(x) = \ell_1(x)f(x_1) + \ell_2(x)f(x_2) = \sin(1) \left[\frac{x(x + 1)}{2} + \frac{x(x - 1)}{2} \right] = \sin(1)x^2 = 0.841x^2. \quad (0.1)$$

Alternatively, you can use Newton's divided difference method to solve (b).

$$\begin{aligned} x_0 &= 0 & f[x_0] &= 0 \\ & & f[x_0, x_1] &= \frac{\sin 1 - 0}{1 - 0} = \sin 1 \\ x_1 &= 1 & f[x_1] &= \sin 1 & f[x_0, x_1, x_2] &= \frac{\sin 1 - 0}{0 + 1} = \sin 1 \\ & & f[x_1, x_2] &= \frac{\sin 1 - \sin 1}{-1 - 1} = 0 \\ x_2 &= -1 & f[x_2] &= \sin 1 \end{aligned}$$

Therefore,

$$\begin{aligned} p_2(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &= \sin(1)x + \sin(1)x(x - 1) \\ &= \sin(1)x^2 = 0.841x^2. \end{aligned}$$