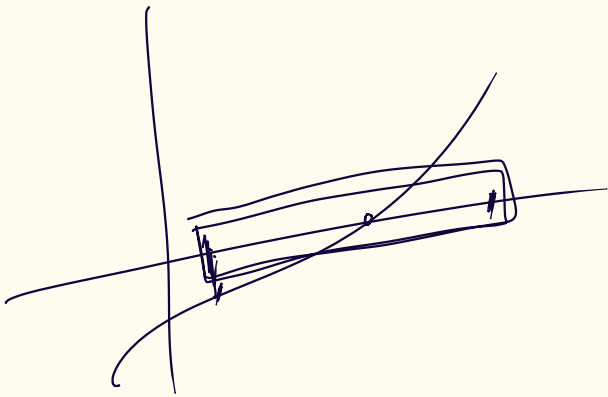


CSE330, 14 March

$$f(x) = 0$$



after n iterations,

$$|m_n - x_*| \leq \frac{|b_0 - a_0|}{\underline{2^{n+1}}}.$$

$$[-100, +100]$$

$$\frac{200}{2^{n+1}} < 10^{-16}$$

$$\Rightarrow \frac{100}{2^n} < 10^{-16}$$

$$\Rightarrow 2^n > 10^{18} \Rightarrow n > \log_2(10^{18}) = 60$$

$$[-1M, +1M]$$

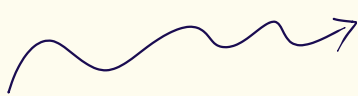
$$\rightarrow n \gg \gg \gg$$

$$f(a) < 0$$

$$f(b) > 0$$

Fixed-point iteration

$$\boxed{\begin{array}{l} f(x) = 0 \\ \text{find } x \\ \text{problem} \end{array}}$$



$$\boxed{\begin{array}{l} g(x) = x \\ \text{find } x \end{array}}$$

fixed-point finding problem.

Suppose, we have to find x s.t.

$$f(x)=0 \quad \boxed{x^2 - 2x - 3 = 0}$$

$$\boxed{3, -1}$$

$$\begin{aligned} x^2 &= 2x + 3 \\ \Rightarrow x &= \boxed{\sqrt{2x+3}} \\ x &= g(x) \\ \boxed{g(x) &= \sqrt{2x+3}} \\ g(x) &= x \end{aligned}$$

$$\begin{aligned} 2x &= x^2 - 3 \\ \Rightarrow x &= \boxed{\frac{x^2 - 3}{2}} \\ &\downarrow \\ &g(x) \end{aligned}$$

$$\begin{aligned} x^2 - 2x &= 3 \\ \Rightarrow x(x-2) &= 3 \\ \Rightarrow x &= \boxed{\frac{3}{x-2}} \\ &g(x) \end{aligned}$$

Algorithm:

- ① choose any $\boxed{x_0} \leftarrow$
- ② $\boxed{x_1} = g(x_0)$
- ③ $x_2 = g(x_1)$
- ④

$\sim 30-40$ iterations.
also depends on x_0 .

Suppose

$$g(x) = \sqrt{2x+3}, \quad x_0 = 0$$

$$x_1 = g(x_0) = \sqrt{3} = 1.73205$$

$$x_2 = g(\quad) = \sqrt{2 \times 1.73205 + 3} = 2.542459$$

$$x_3 = g(\quad) = 2.84339$$

$$x_4 = g(\quad) = 2.9473$$

$$x_5 = g(\quad) = 2.98239$$

Suppose $g(x) = \frac{3}{x-2}; \quad x_0 = 0.$

$$x_1 = -1.5$$

$$x_2 = g(-1.5) = -0.8571$$

$$x_3 = g(\quad) = -1.05$$



How to choose x_0 ?
 " " " $g(x)$? $\rightarrow$ find one that is a contraction map.

Contraction mapping theorem:

Suppose g is a function s.t.

$$\left| \frac{g(x) - g(y)}{x - y} \right| < 1$$

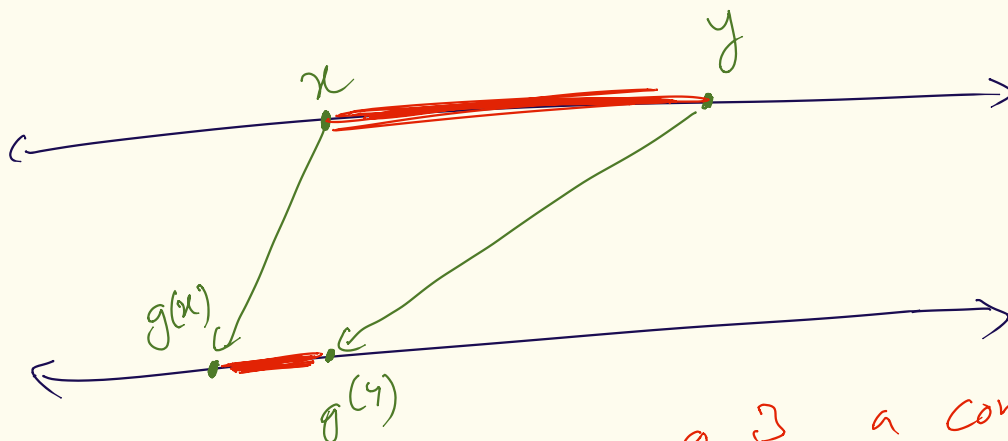
$\forall x, y \in [a, b]$

$\Rightarrow$ then there exists a unique fixed-point.

& given any x_0 , then it converges to a fixed-point.

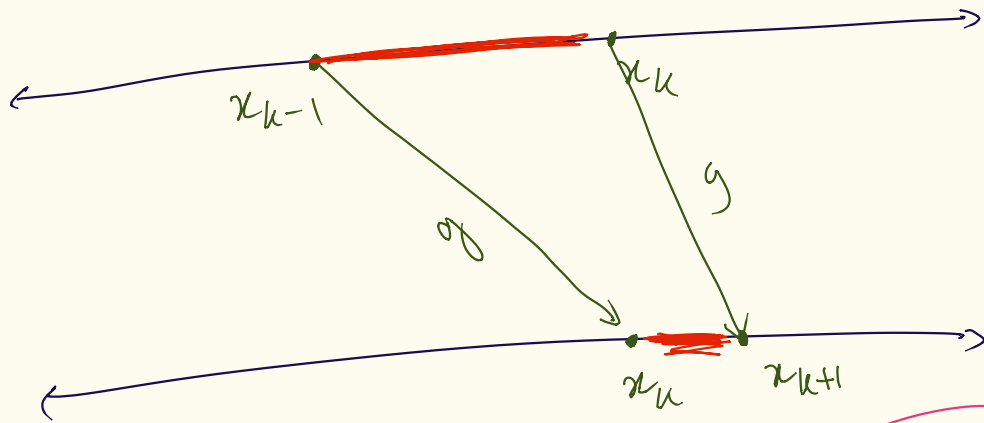
$$\left| \frac{g(x) - g(y)}{x - y} \right| < 1$$

$$\Leftrightarrow |g(x) - g(y)| < |x - y|$$



g is a contraction map.

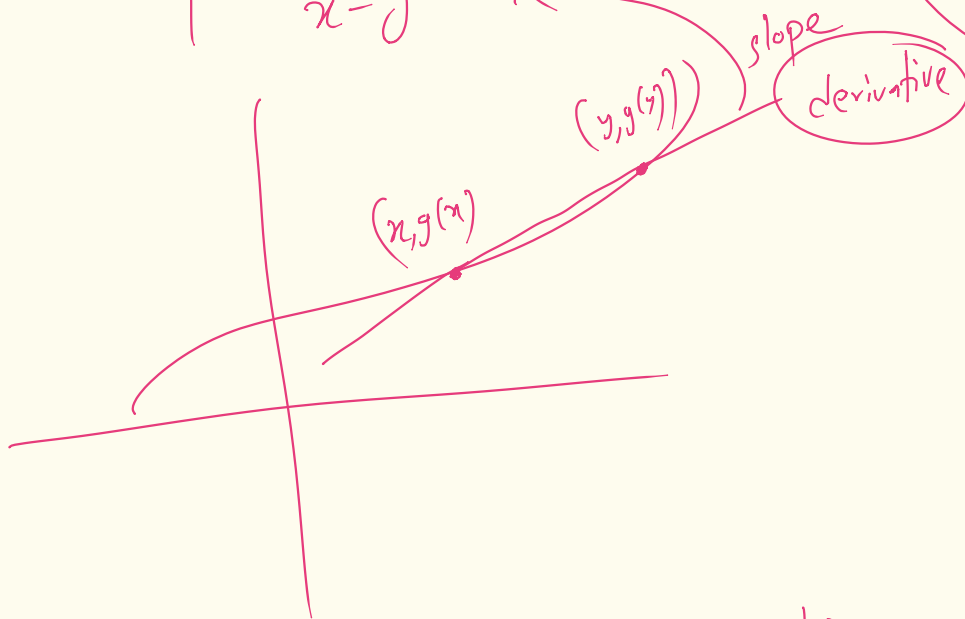
$$x_{k+1} = g(x_k)$$



$$\left| \frac{g(x) - g(y)}{x - y} \right|$$

$$\frac{f(x+h) - f(x)}{h}$$

$$f'(x)$$



To check if g is a contraction map!

$$g' < 1$$

✓

Example:

$$g(x) = \sqrt{2x+3}$$

is it a CM?
(contraction map)

$$g'(x) = \frac{1}{2} \cdot \frac{1}{\sqrt{2x+3}} \cdot 2$$
$$= \frac{1}{\sqrt{2x+3}}$$

< 1

when is it < 1 ?

$$\frac{1}{\sqrt{2x+3}} < 1 \Leftrightarrow \sqrt{2x+3} > 1$$

$$\Leftrightarrow 2x+3 > 1$$

$$\Leftrightarrow x > -1$$

we can choose x_0 any value
 > -1 ,

and g will be CM.

How fast does it converge?

Suppose x_* is a fixed point of g ; i.e. $g(x_*) = x_*$.

• it converges linearly if $|g'(x_*)| < 1$.

• " " superlinearly if $(g'(x_*) = 0)$

$$f(x) = x^2 - 2x - 3 = 0.$$

$$g(x) = \frac{x^2 + 3}{2x - 2}$$

$$g(x) = \sqrt{2x+3}$$

$$\Rightarrow g'(x) = \frac{1}{\sqrt{2x+3}}$$

$$g'(3) = \left(\frac{1}{3}\right)$$

Linearly

$$2x - 2$$

$$x^2 - 2x - 3$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

when $f(x) = 0$, $g(x) = x$

this gives superlinear convergence always.

$$g(x) = \frac{x^2 + 3}{2x - 2}$$

$$= \frac{1}{2} \cdot \frac{x^2 + 3}{x - 1}$$

$$\rightarrow g'(x) = \frac{2x \cdot (x-1) - (x^2 + 3) \cdot 1}{2 \cdot (x-1)^2}$$

$$= \frac{x^2 - 2x - 3}{2(x-1)^2}$$

3, -1

$$g'(3) = 0$$

$$g'(-1) = 0$$

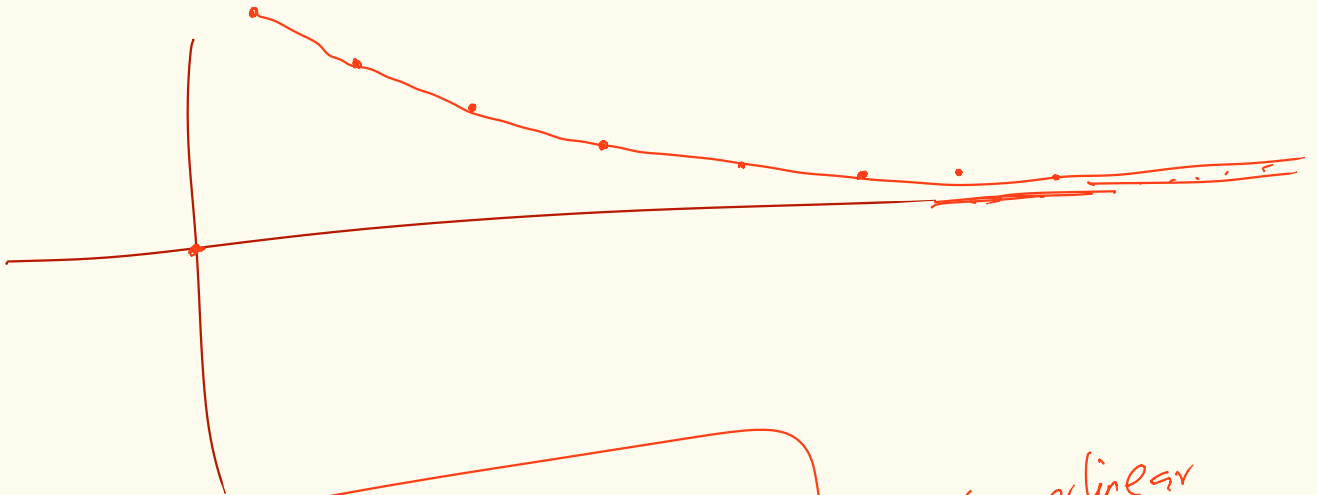
$$x_0,$$

$$x_1 = g(x_0).$$

$$x_2 = g(x_1), \dots$$

$$x_0, x_1, x_2, \dots$$

approaching
convergence



$$g(x) = x - \frac{f(x)}{f'(x)}$$

Newton-Raphson Method

Superlinear

$$x_0 = 0, \quad x_1 = \text{---}, \quad x_2 = \text{---}$$

Linearly

ϵ_M

$$\sqrt{2x+3}$$

$$\frac{x^2+3}{2x-2}$$

33 iteration

5 iteration

Superlinear

