Finite-Dimensional Irreducible Representations of the Weyl Algebra in Characteristic $p > 0$

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What is Weyl Algebra?

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Then we quotient out $k\langle x, y\rangle$ by the ideal generated by the element $yx - xy - 1$.
This is the first Weyl algebra A_1:

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$$A_1 = k\langle x, y\rangle / \langle yx - xy - 1\rangle .$$

Theorem 1

A basis for A_1 is $\left\{x^i y^j \mid i, j \geq 0\right\}$.

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So Weyl Algebra is the algebra generated by position and momentum operators!

We can similarly define the n-th Weyl Algebra A_n by taking the "canonnical commutation relations" of n position and n momentum operators.

$$A_n = \frac{k\left\langle x_1, x_2, \ldots, x_n, y_1 y_2, \ldots, y_n \right\rangle}{\left\langle y_j x_i - x_i y_j - \delta_{ij} \mid 1 \leq i, j \leq n \right\rangle}.$$

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But in this talk, we'll mainly focus on A_1.

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Definition 1

A **representation** of an (associative) algebra A is a vector space V along with a homomorphism $\rho : A \to \mathrm{End}(V)$.

Abuse of notation: for $a \in A$ and $\mathbf{v} \in V$, $\rho(a) : V \to V$ is a linear map, so that $\rho(a)\mathbf{v} \in V$. We shall often write $a\mathbf{v}$ instead of $\rho(a)\mathbf{v}$.

Some more definitions

Definition 2

A **subrepresentation** of a representation V is a subspace $U \subseteq V$ such that $\rho(a)\,U \subseteq U$ for every $a \in A$.

In such a case, we call U **invariant** under the action of A.

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Definition 3

We call a representation V **irreducible** if the only subrepresentations are 0 and V itself.

Some more definitions

Definition 4

Let $\rho_1 : A \to \mathrm{End}(V_1)$ and $\rho_2 : A \to \mathrm{End}(V_2)$ be two representations of A. Then a **homomorphism of representations** is a linear map $\phi : V_1 \to V_2$ such that

$$\phi \circ \rho_1 (a) = \rho_2 (a) \circ \phi,$$

for every $a \in A$. In other words, the following diagram commutes for every $a \in A$:

$$\begin{tikzcd} V_1 \arrow[r, "\phi"] \arrow[d, "\rho_1(a)"'] & V_2 \arrow[d, "\rho_2(a)"] \\ V_1 \arrow[r, "\phi"'] & V_2 \end{tikzcd}$$

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An isomorphism of representations is an invertible homomorphism of representations.

Schur's Lemma

Proposition 2

Let A be an algebra over a field k, and $\phi : V_1 \to V_2$ is a nonzero homomorphism of representations.

1. If V_1 is irreducible, ϕ is injective.
2. If V_2 is irreducible, ϕ is surjective.
3. If both are irreducible, ϕ is an isomorphism.

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Proof.

Just look at $\operatorname{Ker}\phi$ (or $\operatorname{im}\phi$), and they are subrepresentations. So irreducibility forces that $\operatorname{Ker}\phi = 0$ (or $\operatorname{im}\phi = V_2$). ■

Schur's Lemma

Corollary 3

Let A be an algebra over an **algebraically closed** field k, and V is a finite dimensional irrep. If $\phi : V \to V$ is a homomorphism of representations, then $\phi = c\,\mathbb{1}_V$ for some $c \in k$.

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Proof.

If λ is an eigenvalue of ϕ, then $\phi - \lambda\,\mathbb{1}_V$ is a homomorphism of algebras, which is not an isomorphism. So it has to be 0. ■

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Corollary 4

Let A be an algebra over an algebraically closed field k, and V is a finite dimensional irrep. If $a \in Z(A)$, then $\rho(a) = c\,\mathbb{1}_V$ for some $c \in k$.

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Representation of Weyl Algebra in characteristic 0

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By Stone-Von Neumann theorem, every irreducible representation of the position and momentum operators are **unitarily equivalent** to the usual ones: position operator is multiplication by x, momentum operator is $\frac{\partial}{\partial x}$.

So the only irrep of $A_1 = k\langle x, y\rangle / \langle yx - xy - 1\rangle$ looks like

$$\rho(y) = \frac{\mathrm{d}}{\mathrm{d}t}, \quad \rho(x) = t \tag{1}$$

acting on an infinite dimensional space, maybe $k[t]$.

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Let V be a finite dimensional irrep. Since $yx - xy = 1$ in A_1,

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Taking trace on both sides, we get $0 = \dim V$.

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Unless, a positive number is **EQUAL TO** 0.

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Let's now look at the Center of A_1.

Proposition 5

$$\begin{aligned} \left[x^i y^j, x\right] &= j x^i y^{j-1} \\ \left[x^i y^j, y\right] &= -i x^{i-1} y^j \end{aligned} \tag{3}$$

Proof.

Induction on i and j. ∎

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Proof.

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So $k\left[x^p, y^p\right] \in Z(A_1)$. In fact, $Z(A_1) = k\left[x^p, y^p\right]$.

Representation of Weyl Algebra in characteristic $p > 0$

Definition 5

Let $V \neq 0$ be a representation of A. We say that a vector $\mathbf{v} \in V$ is **cyclic** if it generates V, i.e. $A\mathbf{v} = \{\rho(a)\,\mathbf{v} \mid a \in A\} = V$.

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A representation V of A is irreducible if and only if every nonzero vector $\mathbf{v} \in V$ is cyclic.

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A representation V of A is irreducible if and only if every nonzero vector $\mathbf{v} \in V$ is cyclic.

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If $\mathbf{v}$ is not cyclic, then $A\mathbf{v} := \{\rho(a)\,\mathbf{v} \mid a \in A\}$ is a proper subrepresentation. ■

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A representation V of A is irreducible if and only if every nonzero vector $\mathbf{v} \in V$ is cyclic.

Proof.

If $\mathbf{v}$ is not cyclic, then $A\mathbf{v} := \{\rho(a)\,\mathbf{v} \mid a \in A\}$ is a proper subrepresentation. Conversely, if $W \subseteq V$ is a subrepresentation, then for $\mathbf{w} \in W$, $V = A\mathbf{w} \subseteq W$, proving that $W = V$. ∎

Representation of Weyl Algebra in characteristic $p > 0$

$\rho(y^p) = c\,\mathbb{1}_V$ since $y^p \in Z(A_1)$. Let $\mathbf{v}$ be an eigenvector of $\rho(y)$ with the eigenvalue λ (where $\lambda^p = c$). Then $\mathbf{v}$ is cyclic!

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Therefore, $\dim V = p$.

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As stated earlier, $\rho(y)$ has eigenvalue λ, where $\lambda^p = c$. p-th roots are unique in characteristic p, so $\rho(y)$ has just one eigenvalue.

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Say v_1 and v_2 are two eigenvectors for the eigenvalue λ. Then

$$\mathbf{v}_2 = \rho\,(P(x))\,\mathbf{v}_1, \tag{4}$$

for a polynomial $P(x)$ of degree at most $p - 1$.

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for a polynomial $P(x)$ of degree at most $p-1$. Then

$$[y, P(x)] = P'(x). \tag{5}$$

Representation of Weyl Algebra in characteristic $p > 0$

$$\rho\left(P'(x)\right)\mathbf{v}_1 = \rho(y)\rho\left(P(x)\right)\mathbf{v}_1 - \rho\left(P(x)\right)\rho(y)\mathbf{v}_1$$

Representation of Weyl Algebra in characteristic $p > 0$

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Therefore, $\rho\left(P'(x)\right) = 0$, so that $\mathbf{v}_2$ is a constant multiple of $\mathbf{v}_1$.

Therefore, not only does $\rho(y)$ have only one eigenvalue, it has only one linearly independent eigenvector.

Representation of Weyl Algebra in characteristic $p > 0$

Therefore, not only does $\rho(y)$ have only one eigenvalue, it has only one linearly independent eigenvector. So in a suitable basis, it looks like a Jordan block:

$$\rho(y) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix} \tag{6}$$

Representation of Weyl Algebra in characteristic $p > 0$

So all that is left is to compute is $\rho(x)$. But we don't quite know what the basis is yet. So we're gonna stick to the previous basis that we showed earlier:

$$B = \left\{\rho\left(x^i\right)\mathbf{v} \mid 0 \leq i \leq p-1\right\} = \left\{\mathbf{v}, \rho(x)\mathbf{v}, \rho\left(x^2\right)\mathbf{v}, \ldots, \rho\left(x^{p-1}\right)\mathbf{v}\right\}.$$

Representation of Weyl Algebra in characteristic $p > 0$

How $\rho(x)$ acts on these basis vectors is pretty simple. Since $\rho\left(x^p\right)$ is a scalar, say $\rho\left(x^p\right) = \mu \mathbb{1}_V$,

$$\rho(x)\left[\rho\left(x^i\right)\mathbf{v}\right] = \begin{cases} \rho\left(x^{i+1}\right)\mathbf{v} & \text{if } i \neq p-1 \\ \mu\mathbf{v} & \text{if } i = p-1 \end{cases} \tag{7}$$

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So $\rho(x)$ shifts all the basis vectors once to the next, and for the final one, it scales by μ. So the matrix representation of $\rho(x)$ in this (ordered) basis $B = \left\{\mathbf{v}, \rho(x)\mathbf{v}, \rho\left(x^2\right)\mathbf{v}, \ldots, \rho\left(x^{p-1}\right)\mathbf{v}\right\}$ is

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \mu \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{8}$$

Representation of Weyl Algebra in characteristic $p > 0$

Finally, we just need to compute $\rho(y)$'s action on these basis vectors. Since $\left[y, x^i\right] = ix^{i-1}$,

$$\begin{aligned} \rho(y)\left[\rho\left(x^i\right)\mathbf{v}\right] &= \rho\left(\left[y, x^i\right]\right)\mathbf{v} + \rho\left(x^i\right)\rho(y)\mathbf{v} \\ &= i\rho\left(x^{i-1}\right)\mathbf{v} + \lambda\rho\left(x^i\right)\mathbf{v}. \end{aligned} \tag{9}$$

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Representation of Weyl Algebra in characteristic $p > 0$

So in characteristic p, the irreducible representations of A_1 are p-dimensional, and they look like

$$\rho(x) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \mu \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad \rho(y) = \lambda \mathbb{1} + \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & p-1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Looks familiar?

Consider the space $\mathbb{C}[t]$ of all polynomials over $\mathbb{C}$. In this infinite dimensional space, the matrix representation of $\frac{\mathrm{d}}{\mathrm{d}t}$ in the ordered basis $\{1, t, t^2, t^3, \dots\}$ looks like this

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Consider the space $\mathbb{C}[t]$ of all polynomials over $\mathbb{C}$. In this infinite dimensional space, the matrix representation of $\frac{\mathrm{d}}{\mathrm{d}t}$ in the ordered basis $\{1, t, t^2, t^3, \dots\}$ looks like this

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and the "matrix" representation of multiplication by t looks like this

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Representation of Weyl Algebra in characteristic $p > 0$

$$\rho(x) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \mu \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{11}$$

So $\rho(x)$ looks exactly like multiplication by t on $k[t]$, except for t^p being identified with the scalar μ.

Representation of Weyl Algebra in characteristic $p > 0$

$$\rho(x) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \mu \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{11}$$

So $\rho(x)$ looks exactly like multiplication by t on $k[t]$, except for t^p being identified with the scalar μ. So our representation space is gonna be $k[t] / \langle t^p - \mu \rangle$.

Representation of Weyl Algebra in characteristic $p > 0$

$$\rho(y) = \lambda \mathbb{1} + \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & p-1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{12}$$

And $\rho(y)$ looks exactly like λ plus $\frac{\mathrm{d}}{\mathrm{d}t}$ on $k[t] / \langle t^p - \mu \rangle$.

Representation of Weyl Algebra in characteristic $p > 0$

To summarize, the irreducible representations of A_1 in characteristic p are on the representation space $k\,[t]\,/\,\langle t^p - \mu \rangle$, where

$$\begin{aligned} \rho(x) &= \text{multiplication by } t \\ \rho(y) &= \lambda + \frac{\mathrm{d}}{\mathrm{d}t} \end{aligned} \tag{13}$$

for $\lambda, \mu \in k$.

Concluding Remarks

So, in finite dimensions and char $k = p > 0$ as well, the representation of Weyl Algebra looks exactly like those for $k = \mathbb{C}$.

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What's next? Defining Quantum Mechanics over an algebraically closed field k with char $k = p > 0$?

Concluding Remarks

So, in finite dimensions and char $k = p > 0$ as well, the representation of Weyl Algebra looks exactly like those for $k = \mathbb{C}$.

What's next? Defining Quantum Mechanics over an algebraically closed field k with char $k = p > 0$?

Apparently some people are trying to do that! And it might have some cool applications in Quantum Computing! See references!

References

1. *Introduction to Representation Theory*, by Etingof et al.
2. *Categorical Quantum Computing with Finite Fields*, by Matthew Varughese. `https://www.cs.ox.ac.uk/people/bob.coecke/Varughese.pdf`
3. *Quantum Computing over Finite Fields*, by R.P. James et al. `https://arxiv.org/pdf/1101.3764`
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Thank you for joining!

The slides are available in my webpage
`https://atonurc.github.io/assets/weyl_talk.pdf`