BRAC UNIVERSITY

Inspiring Excellence

Algebraic Topology II (MAT432)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Algebraic Topology II (MAT432)** in Fall 2021 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The recorded video lectures can be found here. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at `atonuroychowdhury@gmail.com`

Atonu Roy Chowdhury

References:

- *Elements of Algebraic Topology*, by James R. Munkres
- *Foundations of Algebraic Topology*, by Samuel Eilenberg & Norman E. Steenrod
- *Axiomatic Approach to Homology Theory*, by Samuel Eilenberg & Norman E. Steenrod. Link to the paper: `https://www.pnas.org/content/pnas/31/4/117.full.pdf`
- *Topology Lecture Notes*, by Cornelia Drutu & Marc Lackenby. Link: `https://www.maths.ox.ac.uk/system/files/attachments/toplectnotes17.pdf`

Contents

1 Lecture 1

Let's recall some concepts from Algebraic Topology I. Suppose K is a finite complex. If we take the set $C_p(K)$ of all p-chains on K, then this set has a free abelian group structure. The elementary p-chains corresponding to the oriented p-simplices form a basis of $C_p(K)$. Since K is finite complex, there are finitely many oriented p-simplices. Hence, $C_p(K)$ is of finite rank.

If we take the boundary homomorphism $\partial_p : C_p(K) \to C_{p-1}(K)$, the kernel of ∂_p is a subgroup of $C_p(K)$. It is known as the group of p-cycles, and we write it $Z_p(K)$. Since $C_p(K)$ is free abelian of finite rank, so is $Z_p(K)$.

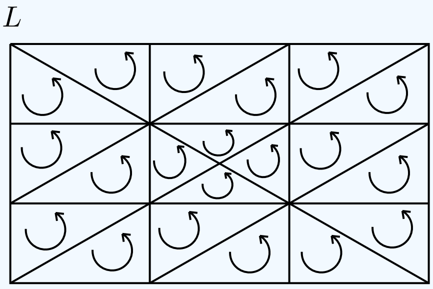
The image of ∂_{p+1} is also a subgroup of $C_p(K)$. It is known as the group of p-boundaries, and we write it $B_p(K)$. The p-th homology group is defined as $H_p(K) = Z_p(K)/B_p(K)$.

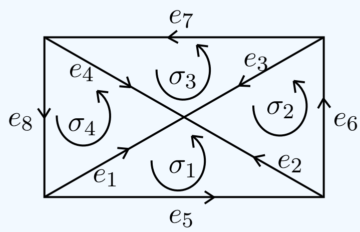
Since $Z_p(K)$ is free abelian of finite rank, $H_p(K)$ is finitely generated. So we can apply *Fundamental theorem of finitely generated abelian group* on $H_p(K)$. The betti number and torsion coefficients of $H_p(K)$ are called, classically, the **betti number** and **torsion coefficients** of K in dimension p.

Let's first prove a lemma that will help us to compute homology groups of some compact surfaces.

Lemma 1.0.1

Let L be the complex shown in the figure below (left) whose underlying space is a rectangle.



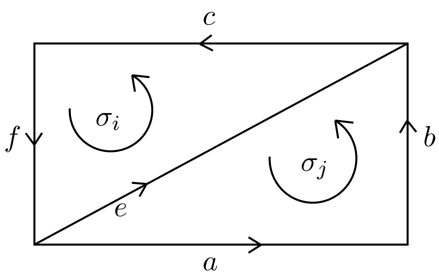


Let Bd L denote the complex whose underlying space is the boundary of the recatngle. We orient each 2-simplex σ_i of L in the counterclockwise direction, and orient the 1-simplices arbitrarily. One possible orientation of the central rectangle is shown on the right. Then the following statements hold:

(i) Every 1-cycle of L is homologous to a 1-cycle carried by Bd L.

(ii) If d is a 2-chain of L and $\partial_2 d$ is carried by Bd L, then d is a multiple of the 2-chain $\sum \sigma_i$.

Proof. We prove (ii) first. Let σ_i and σ_j be two simplices with a common edge e. We know that, for a given 2-chain d, $\partial_2 d$ is carried by Bd L. And the 1-simplices of Bd L is not shared by two 2-simplices. So $\partial_2 d$ is 0 on e.



We orient the four 1-simplices (a, b, c, f) arbitrarily. Suppose $d = \sum n_k \sigma_k$. So, the value of d on the oriented 2-simplies σ_i, σ_j are n_i, n_j respectively. From the figure, we get that

$$\partial_2 \sigma_i = c + f + e \quad \text{and} \quad \partial_2 \sigma_j = a + b - e$$

Therefore, the value of $\partial_2 d$ on e becomes $n_i - n_j$. But we know that the value of $\partial_2 d$ on e is 0. Therefore, $n_i = n_j$.

Similarly, the value of d on all 2-simplices are the same. In other words, $n_k = n$ for all k for some fixed n. Hence, $d = n \sum \sigma_i$.

Let's prove (i) now. We orient the 1-simplices in such a way that the central rectangle looks like the one given on the right of the first figure. Now, let c be a 1-chain of L given by

$$c = \sum_{i \in S} n_i e_i + \sum_{i \notin S} n_i e_i \ , \quad \text{where } S = \{1, 2, \ldots, 8\}$$

The value of c on the 1-simplex e_1 is n_1. Now we define a 1-chain c_1' by $c_1' = c + \partial_2 (n_1 \sigma_1)$. As a result,

$$c_1' = \sum_{i \in S} n_i e_i + \sum_{i \notin S} n_i e_i + n_1 (e_5 + e_2 - e_1) = \sum_{i \in S \setminus \{1\}} n_i' e_i + \sum_{i \notin S} n_i' e_i$$

Therefore, c_1' is homologous to c and the value of c_1' on e_1 is 0.

The value of c_1' on the 1-simplex e_1 is n_2'. Now we define a 1-chain c_1'' by $c_1'' = c_1' + \partial_2 (n_2' \sigma_2)$. As a result,

$$c_1'' = \sum_{i \in S \setminus \{1\}} n_i' e_i + \sum_{i \notin S} n_i' e_i + n_2' (e_6 + e_3 - e_2) = \sum_{i \in S \setminus \{1,2\}} n_i'' e_i + \sum_{i \notin S} n_i'' e_i$$

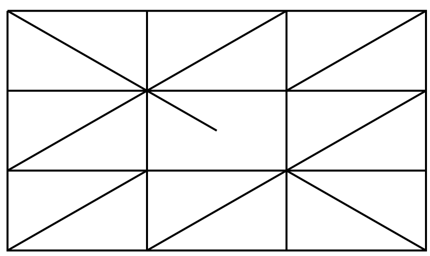
Therefore, c_1'' is homologous to c_1' and the value of c_1'' on e_2 is 0.

The value of c_1'' on the 1-simplex e_3 is n_3''. Now we define a 1-chain c_1 by $c_1 = c_1'' + \partial_2 (n_3'' \sigma_3)$. As a result,

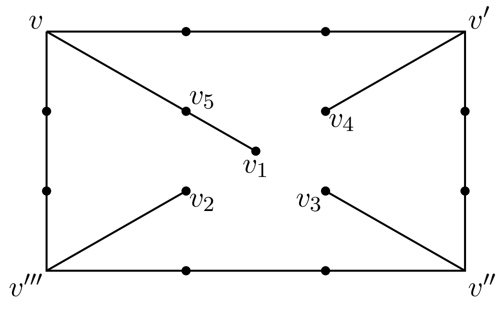
$$c_1 = \sum_{i \in S \setminus \{1,2\}} n_i'' e_i + \sum_{i \notin S} n_i'' e_i + n_3'' (e_7 + e_4 - e_3) = \sum_{i \in S \setminus \{1,2,3\}} n_i''' e_i + \sum_{i \notin S} n_i''' e_i$$

Therefore, c_1 is homologous to c_1'' and the value of c_1 on e_3 is 0.

So we get that $c_1 = c + \partial_2 (n_1 \sigma_1 + n_2' \sigma_2 + n_3 \sigma_3'')$. Therefore, c_1 is homologous to c. Also, c_1 vanishes on the 1-simplices e_1, e_2, e_3. So the 1-chain c is homologous to a 1-chain c_1 carried by the following subcomplex of L:



Using the same method, one can find a 1-chain c_2 that is homologous to c and carried by the following subcomplex of L:



Now, if the given 1-chain c is a 1-cycle, then c_2 is also a 1-cycle. But in order for c_2 to be a 1-cycle, $\partial_1 c_2$ must vanish on the vertices v_1, v_2, v_3, v_4, v_5.

Suppose c_2 is a 1-cycle and its value on the 1-simplices $[v, v_5], [v_5, v_1], [v', v_4], [v'', v_3], [v''', v_2]$ are n_1, n_2, n_3, n_4, n_5 respectively. Since c_2 is a 1-cycle, $\partial_1 c_2 = 0$.

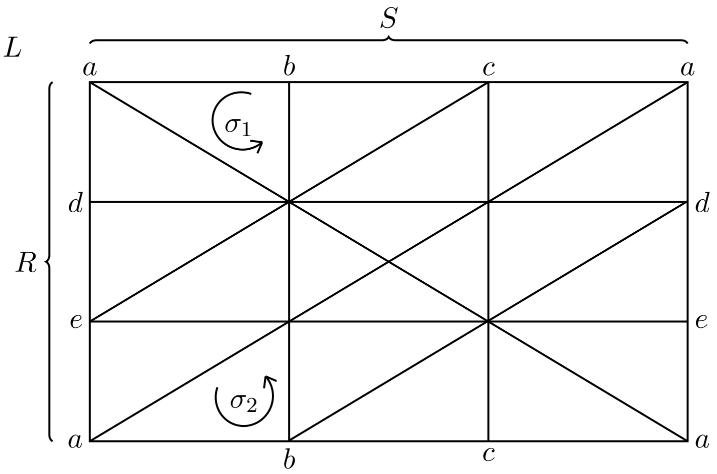
But the value of $\partial_1 c_2$ on v_5 is $n_1 - n_2$; on v_1 is n_2; on v_2 is n_5; on v_3 is n_4; on v_4 is n_3. Since $\partial_1 c_2$ vanishes in all those vertices,

$$n_1 = n_2 \ , \ n_2 = 0 \ , \ n_5 = 0 \ , \ n_4 = 0 \ , \ n_3 = 0 \implies n_1 = n_2 = n_3 = n_4 = n_5 = 0$$

It proves that the 1-cycle c_2 is carried by the boundary of L, *i.e.* $\operatorname{Bd} L$. Therefore, for each 1-cycle c, we can find another 1-cycle c_2 which is homologous to c and carried by $\operatorname{Bd} L$. ■

§1.1 Homology Groups of Torus

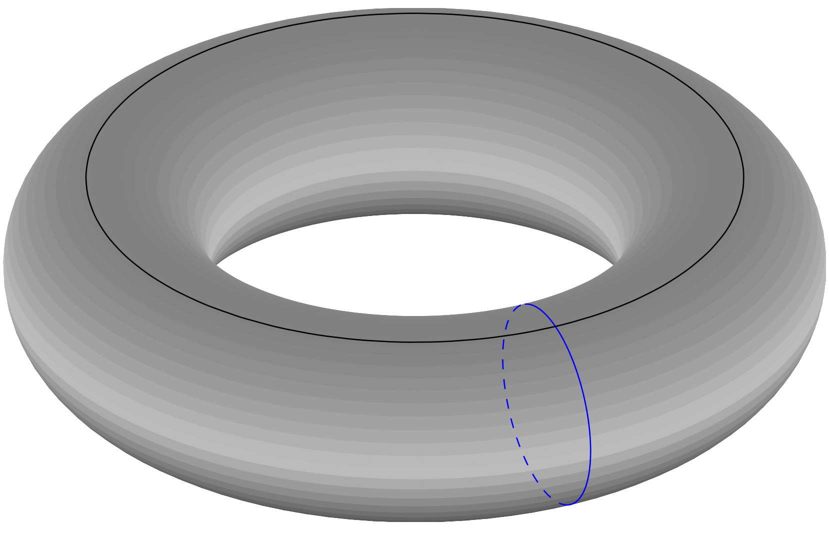
Consider the following labelled rectangle L of the following figure:



The top edge, or equivalently the bottom edge, of L is denoted by S. The left edge, or equivalently the right edge, of L is denoted by R.

Let T denote the complex represented by L. It's underlying space is the torus.

Suppose $g : |L| \to |T|$ is the pasting map. In other words, it glues the left edge of L with the right edge of L; and it glues the top edge of L with the bottom edge. Then we get a torus.

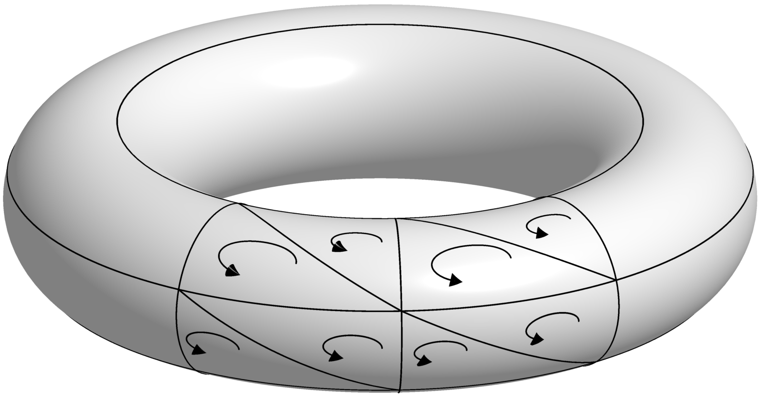


Let $A = g(|\operatorname{Bd} L|)$. Then A is homeomorphic to space that is the union of two circles with a point in common. Such a space is called **wedge of two circles**. In the figure, the blue circle is $g(|R|)$ and the black circle is $g(|S|)$.

We orient all 2-simplices of L counterclockwise. Let $\gamma = \sum \sigma_i$ be the sum of all 2-simplices of L. Also, we orient the 1-simplices arbitrarily.

$$w_1 := [a,b] + [b,c] + [c,a] \quad \text{and} \quad z_1 := [a,d] + [d,e] + [e,a]$$

The gluing map g preserves orientation of the simplices. Therefore, the simplices of T behave exactly the same way as those of L.



Since g makes identifications only among simplices of $\operatorname{Bd} L$, the arguments we gave earlier in proving Lemma 1.0.1 still applies. Therefore, we have

(i) Every 1-cycle of T is homologous to a 1-cycle carried by A.

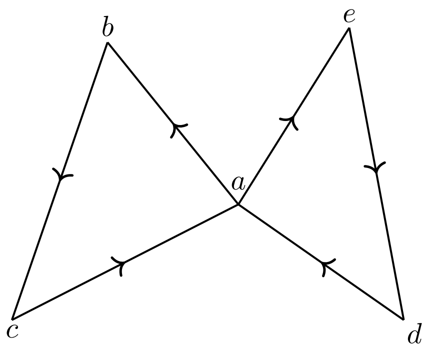
(ii) If d is a 2-chain of T and $\partial_2 d$ is carried by A, then d is a multiple of γ.

In addition to these two results, T also satisfies some more properties.

Proposition 1.1.1

If c is a 1-cycle of T carried by A, then c is of the form $nw_1 + mz_1$.

Proof. Note that $A = g(|\operatorname{Bd} L|)$, being the wedge sum of two circles, is actually the 1-dimensional complex pictured below:



Let c be a 1-cycle carried by the above complex.

$$c = n_1[a,b] + n_2[b,c] + n_3[c,a] + n_4[a,e] + n_5[e,d] + n_6[d,a]$$

Since c is a 1-cycle, $\partial_1 c = 0$.

$$\begin{aligned} 0 = \partial_1 c &= n_1(b-a) + n_2(c-b) + n_3(a-c) + n_4(e-a) + n_5(d-e) + n_6(a-d) \\ &= a(-n_1 + n_3 - n_4 + n_6) + b(n_1 - n_2) + c(n_2 - n_3) + d(n_5 - n_6) + e(n_4 - n_5) \end{aligned}$$

Equating both sides, we get

$$-n_1 + n_3 - n_4 + n_6 = n_1 - n_2 = n_2 - n_3 = n_5 - n_6 = n_4 - n_5 = 0$$

$$\therefore n_1 = n_2 = n_3 \quad \text{and} \quad n_4 = n_5 = n_6$$

Therefore,

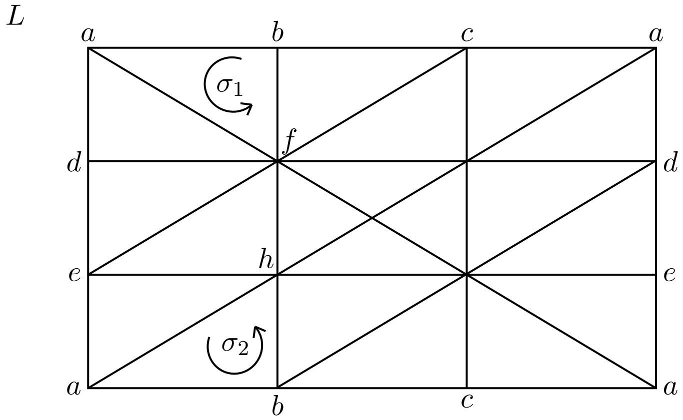
$$c = n_1 \left([a,b] + [b,c] + [c,a] \right) - n_4 \left([a,d] + [d,e] + [e,a] \right) = n w_1 + m z_1$$

where $n = n_1$ and $m = -n_4$. ■

Proposition 1.1.2

$\partial_2 \gamma = 0$.

Proof. Let's look back at the complex L again:



Recall that all the 2-simplices σ_i are oriented counterclockwise. Then

$$\partial_2 \sigma_1 = [b,a] + [a,f] + [f,b] \quad \text{and} \quad \partial_2 \sigma_2 = [a,b] + [b,h] + [h,a]$$

Therefore, in $\partial_2 \sigma_1 + \partial_2 \sigma_2$ the contribution of $[a,b]$ is 0. In other words, $\partial_2 \gamma = \sum \partial_2 \sigma_i$ has the value 0 on the 1-simplex $[a,b]$.

Similarly, one can show that $\partial_2 \gamma$ vanishes on all 1-simplices of $\operatorname{Bd} L$. Also, one can explicitly show that $\partial_2 \gamma$ vanishes on all 1-simplices not in $\operatorname{Bd} L$. Therefore, $\partial_2 \gamma$. ■

Now we can compute the holomogy groups of T.

Theorem 1.1.3

$H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H_2(T) \cong \mathbb{Z}$. Furthermore, γ generates $H_2(T)$; w_1 and z_1 represent a basis for $H_1(T)$.

Proof. Let's take a 1-cycle c in T. By Property (i), c is homologous to a 1-cycle c_1 carried by A. By Proposition 1.1.1, $c_1 = n w_1 + m z_1$ for some $n, m \in \mathbb{Z}$. Therefore, $Z_1(T)$ is a free group generated by w_1 and z_1. Hence, $Z_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Now, such a cycle bounds if there exists a 2-chain d of T such that $c_1 = \partial_2 d$. Since c_1 carried by A, d is also carried by A. Then by Property (ii), $d = p\gamma$ for some $p \in \mathbb{Z}$. Now, by Proposition 1.1.2,

$$c_1 = \partial_2 d = \partial_2 (p\gamma) = p\, \partial_2 \gamma = 0$$

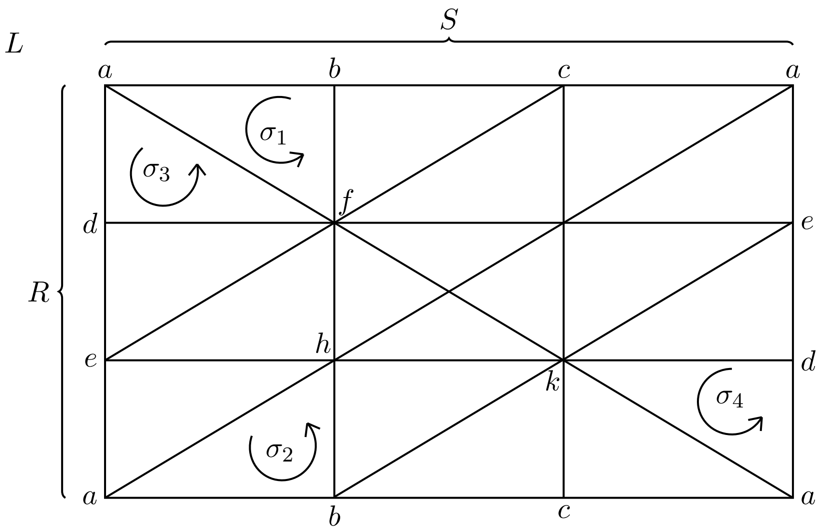
Therefore, $B_1(T) = 0$. As a result, $H_1(T) = Z_1(T) / B_1(T) = Z_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$. And it is generated by w_1 and z_1.

Let us compute $H_2(T)$ now. Let d be a 2-cycle of T. Then $\partial_2 d = 0$. In particular, $\partial_2 d$ is 0 on every 1-simplex not in A. Thus $\partial_2 d$ is carried by A. Therefore, by Property (ii), $d = p\gamma$ for some $p \in \mathbb{Z}$. Hence, $Z_2(T)$ is a free group generated by γ. So $Z_2(T) \cong \mathbb{Z}$.

There are no 3-chains in T. So ∂_3 is a trivial homomorphism. As a result, $B_2(T) = 0$. Hence, $H_2(T) = Z_2(T) / B_2(T) = Z_2(T) \cong \mathbb{Z}$. And it is generated by γ. ■

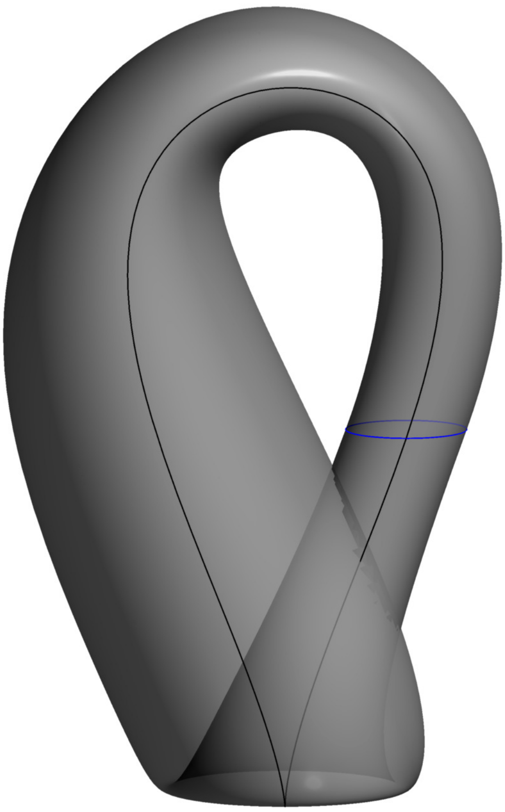
§1.2 Homology Groups of Klein Bottle

Consider the following labelled rectangle L of the following figure:



The top edge of L is denoted by S. The left edge of L is denoted by R.

Let K denote the complex represented by L. Suppose $g : |L| \to |K|$ is the pasting map. Then K is a klein bottle



Let $A = g(|\operatorname{Bd} L|)$. Then A is homeomorphic to a wedge of two circles. In the figure, the blue circle is $g(|R|)$ and the black loop is $g(|S|)$.

We orient all 2-simplices of L counterclockwise. Let $\gamma = \sum \sigma_i$ be the sum of all 2-simplices of L. Also, we orient the 1-simplices arbitrarily.

$$w_1 := [a, b] + [b, c] + [c, a] \quad \text{and} \quad z_1 := [a, d] + [d, e] + [e, a]$$

Since g makes identifications only among simplices of $\operatorname{Bd} L$, the arguments we gave earlier in proving Lemma 1.0.1 still applies. Therefore, we have

(i) Every 1-cycle of K is homologous to a 1-cycle carried by A.

(ii) If d is a 2-chain of K and $\partial_2 d$ is carried by A, then d is a multiple of γ.

In addition to these two results, K also satisfies some more properties. Since A is a wedge of two circles, Proposition 1.1.1 holds for klein bottle too.

Proposition 1.2.1

$\partial_2 \gamma = 2z_1$.

Proof. According to the picture of the complex L,

$$\partial_2 \sigma_1 = [b, a] + [a, f] + [f, b] \quad \text{and} \quad \partial_2 \sigma_2 = -[b, a] + [b, h] + [h, a]$$

Therefore, the value of $\partial_2 \gamma$ on $[a, b]$ is 0. Similarly, one can verify that $\partial_2 \gamma$ vanishes on $[b, c]$ and $[c, a]$ too. Similar to the case of torus, $\partial_2 \gamma$ vanishes on all the 1-simplices not belonging to $\operatorname{Bd} L$. Now,

$$\partial_2 \sigma_3 = [a, d] + [d, f] + [f, a] \quad \text{and} \quad \partial_2 \sigma_4 = [a, d] + [d, k] + [k, a]$$

Therefore, the value of $\partial_2 \gamma$ on $[a, d]$ is 2. Similarly, the value of $\partial_2 \gamma$ on $[d, e]$ and $[e, a]$ are both 2.

Hence, $\partial_2 \gamma = 2[a, d] + 2[d, e] + 2[e, a] = 2z_1$. ■

Now we are ready to compute the homology groups of K.

Theorem 1.2.2

$H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and $H_2(K) = 0$. Furthermore, the torsion element of $H_1(K)$ is represented by the chain z_1, and a generator for the group $H_1(K)$ modulo torsion is represented by w_1.

Proof. Let's take a cycle c in K. By Property (i) and Proposition 1.1.1, c is homologous to a 1-cycle c_1 of the form $c_1 = nw_1 + mz_1$. Therefore, $Z_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}$.

If the 1-cycle c_1 also bounds, then $c_1 = \partial_2 d$ for some 2-chain d of K. By Property (ii), since c_1 is carried by A, $d = p\gamma$ for some $p \in \mathbb{Z}$. Hence,

$$c_1 = \partial_2 d = \partial_2 (p\gamma) = p\, \partial_2 \gamma = 2pz_1$$

Therefore, the 1-cycle $c_1 = nw_1 + mz_1$ bounds if and only if $n = 0$ and m is even. A generic element of $B_1(K)$ is $(0, 2p)$ for $p \in \mathbb{Z}$. Therefore, $B_1(K) \cong 2\mathbb{Z}$.

Now, $H_1(K) = Z_1(K)/B_1(K)$ contains equivalence classes of ordered pairs of integers under the following equivalence relation $\sim$:

$$\text{for } (m, n), (m', n') \in \mathbb{Z} \oplus \mathbb{Z}\,,\ (m, n) \sim (m', n') \text{ if } m = m' \text{ and } 2 \mid n - n'$$

So, the equivalence classes in $H_1(K)$ look like $[(a, 0)]$ or $[(b, 1)]$. As a result, $H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. Here z_1 represents the torsion element of $H_1(K)$, while w_1 generates $H_1(K)/T_1(K)$ where $T_1(K)$ is the torsion subgroup of $H_1(K)$.

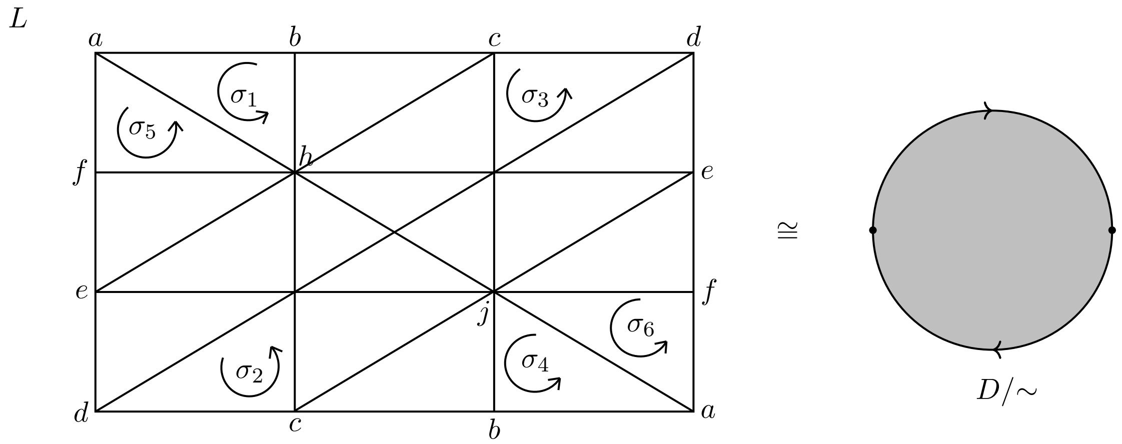
Let's now compute $H_2(K)$. Let d be a 2-cycle of K. Then $\partial_2 d = 0$. Since $\partial_2 d$ vanished on every 1-simplex not belonging in A, by Property (ii), $d = p\gamma$ for some $p \in \mathbb{Z}$. If $p \neq 0$,

$$\partial_2 d = \partial_2 (p\gamma) = p\, \partial_2 \gamma = 2pz_1 \neq 0$$

Therefore, $p = 0$ and hence $d = 0$. This concludes that the group $Z_2(K)$ of 2-cycles is trivial. As a result, $H_2(K) = Z_2(K)/B_2(K)$ is also trivial, *i.e.* $H_2(K) = 0$. ■

§1.3 Homology Groups of Projective Plane

Consider the following labelled rectangle L of the following figure:



Let P^2 denote the complex represented by L. The underlying space $|P^2|$ of the complex P^2 (after identifications are made along the edges obeying the labels) is the projective plane. Suppoe $g : |L| \to |P^2|$ is the corresponding gluing map.

Proposition 1.3.1

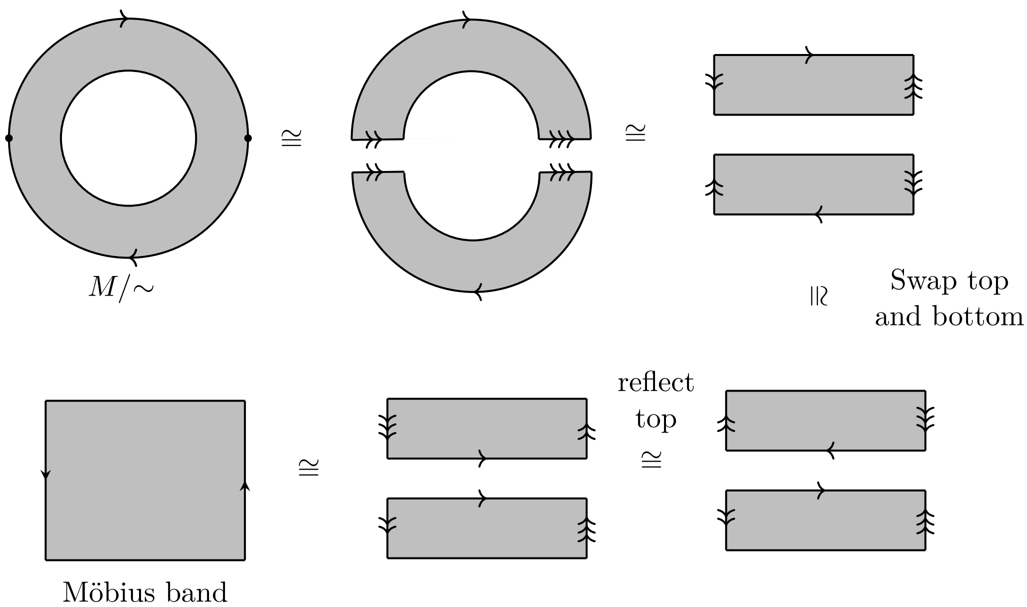
$A = g(|\text{Bd}\, L|)$ is homeomorphic to a circle.

Proof. It is immediate that the labelled complex L, whose underlying space is a rectangle, is homeomorphic to a 2-disk with the prescribed identification along the boundary (diametrically opposite points of the disk are meant to be glued by the gluing map g).

$D/{\sim}$ is the 2-disk with antipodal points identified on the boundary. Now, first we decompose D into an annulus M and a smaller 2-disk so that attaching the inner circle of M along the boundary of the smaller disk gives the full 2-disk D. Hence attaching a 2-disk to $M/{\sim}$ should give $D/{\sim} \cong P^2$.

Now, if we can show that $M/{\sim}$ is homeomorphic to a Möbius band (whose boundary is homeomorphic to a circle), we'll be able to show that A is homeomorphic to a circle; and the real projective plane P^2 is the topological space one obtains by gluing the boundary of a Möbius band along the boundary of a 2-disk.

The visual proof is presented below:



Hence, $M/\sim$ is homeomorphic to a Möbius band. Therefore, $A = g(|\mathrm{Bd}\, L|)$ is homeomorphic to the boundary of a a Möbius band. Thus $A \cong S^1$. ■

We orient all 2-simplices of L counterclockwise. Let $\gamma = \sum \sigma_i$ be the sum of all 2-simplices of L. Also, we orient the 1-simplices arbitrarily.

$$z_1 := [a, b] + [b, c] + [c, d] + [d, e] + [e, f] + [f, a]$$

Since g makes identifications only among simplices of $\mathrm{Bd}\, L$, the arguments we gave earlier in proving Lemma 1.0.1 still applies. Therefore, we have

(i) Every 1-cycle of P^2 is homologous to a 1-cycle carried by A.

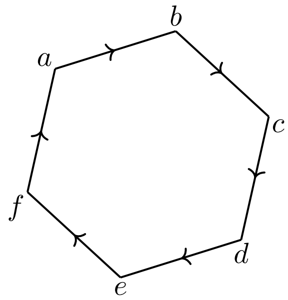
(ii) If d is a 2-chain of P^2 and $\partial_2 d$ is carried by A, then d is a multiple of γ.

In addition to these two results, P^2 also satisfies some more properties.

Proposition 1.3.2

Every 1-cycle carried by A is a multiple of z_1.

Proof. Since A is homeomorphic to a circle, we can picture it as the following complex:



Let c be a 1-cycle carried by A.

$$c = n_1 [a, b] + n_2 [b, c] + n_3 [c, d] + n_4 [d, e] + n_5 [e, f] + n_6 [f, a]$$

Since c is a 1-cycle, $\partial_1 c = 0$.

$$\begin{aligned} 0 = \partial_1 c &= n_1 (b - a) + n_2 (c - b) + n_3 (d - c) + n_4 (e - d) + n_5 (f - e) + n_6 (a - f) \\ &= a (n_6 - n_1) + b (n_1 - n_2) + c (n_2 - n_3) + d (n_3 - n_4) + e (n_4 - n_5) + f (n_5 - n_6) \end{aligned}$$

Equating both sides, we get

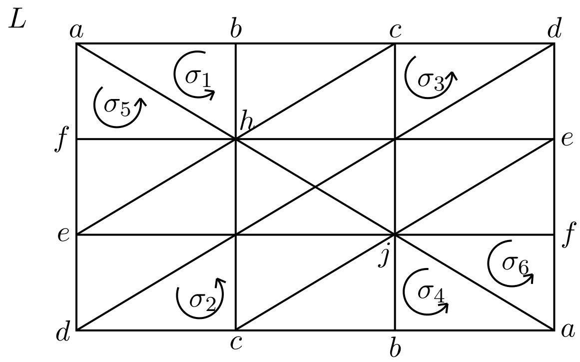
$$n_6 - n_1 = n_1 - n_2 = n_2 - n_3 = n_3 - n_4 = n_4 - n_5 = n_5 - n_6 = 0$$

Therefore, $n_1 = n_2 = n_3 = n_4 = n_5 = n_6$ and thus $c = n_1 z_1$. ■

Proposition 1.3.3

$\partial_2 \gamma = -2z_1$.

Proof. Let's look back at the complex L again:



Recall that all the 2-simplices are oriented counterclockwise.

$$\partial_2\sigma_1 = [b,a] + [a,h] + [h,b] \quad \text{and} \quad \partial_2\sigma_4 = [b,a] + [a,j] + [j,b]$$

Therefore, the value of $\partial_2\sigma_1 + \partial_2\sigma_4$ in $[a,b]$ is -2. Thus the value of $\partial_2\gamma$ on the oriented 1-simplex $[a,b]$ is -2. Similarly,

$$\partial_2\sigma_5 = -[f,a] + [f,h] + [h,a] \quad \text{and} \quad \partial_2\sigma_6 = -[f,a] + [f,j] + [j,a]$$

Therefore, the value of $\partial_2\sigma_5 + \partial_2\sigma_6$ in $[f,a]$ is -2. Thus the value of $\partial_2\gamma$ on the oriented 1-simplex $[f,a]$ is -2.

Similarly, the value of $\partial_2\gamma$ on other oriented 1-simplices of z_1 is -2. Similar as before, $\partial_2\gamma$ vanishes on every 1-simplex not belonging to $\operatorname{Bd} L$. Therefore, $\partial_2\gamma = -2z_1$. ■

Theorem 1.3.4

$H_1\left(P^2\right) \cong \mathbb{Z}_2$ and $H_2\left(P\right) = 0$.

Proof. Let's take a 1-cycle c in P^2. By Property (i) and Proposition 1.3.2, c is homologous to a 1-cycle c_1 of the form $c_1 = nz_1$. So $Z_1\left(P^2\right) \cong \mathbb{Z}$.

If the 1-cycle bounds, then $c_1 = \partial_2 d$ for some 2 chain in P^2. c_1 is carried by A. Hence, by Property (ii), $d = p\gamma$ for some $p \in \mathbb{Z}$. Therefore,

$$c_1 = \partial_2 d = \partial_2\left(p\gamma\right) = p\,\partial_2\gamma = -2pz_1$$

Hence, an element of $B_1\left(P^2\right)$ is an even multiple of z_1. Now we set two 1-cycles c and c' of P^2 to be equivalent if $c - c'$ is an even multiple of z_1.

In other words, all the elements of $Z_1\left(P^2\right)$ are divided into two classes: odd multiples of z_1 and even multiples of z_1. The class of even multiples of z_1 correspond to the identity element of $Z_1\left(P^2\right)/B_1\left(P^2\right) = H_1\left(P^2\right)$; and the class of odd multiples of z_1 correspond to the non-identity element of $H_1\left(P^2\right)$. Hence, $H_1\left(P^2\right) \cong \mathbb{Z}_2$.

Let's now compute $H_2\left(P^2\right)$. Let d be a 2-cycle of P^2. Then $\partial_2 d = 0$. Since $\partial_2 d$ vanished on every 1-simplex not belonging in A, by Property (ii), $d = p\gamma$ for some $p \in \mathbb{Z}$. If $p \neq 0$,

$$\partial_2 d = \partial_2\left(p\gamma\right) = p\,\partial_2\gamma = -2pz_1 \neq 0$$

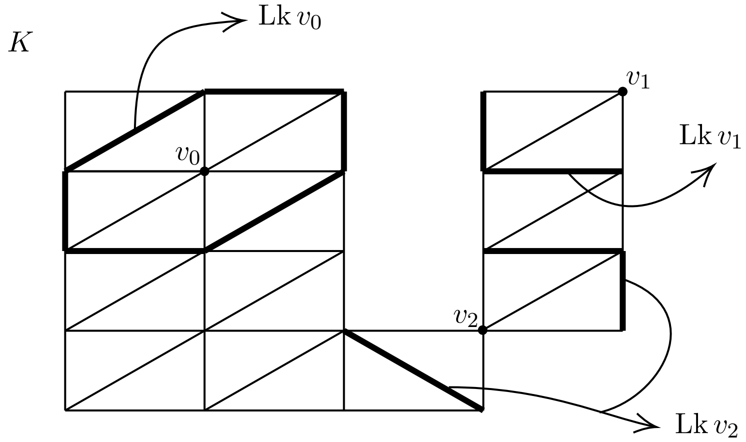
Therefore, $p = 0$ and hence $d = 0$. This concludes that the group $Z_2\left(P^2\right)$ of 2-cycles is trivial. As a result, $H_2\left(P^2\right) = Z_2\left(P^2\right)/B_2\left(P^2\right)$ is also trivial, *i.e.* $H_2\left(P^2\right) = 0$ ■

2 Lecture 2

§2.1 Zero Dimensional Homology

Definition 2.1.1 (Star and Link). Let v be a vertex of a simplicial complex K. The **star** of v, denoted by $\operatorname{St} v$, is the union of the interior of all the simplices that have v as a vertex. The closure of $\operatorname{St} v$, denoted by $\overline{\operatorname{St}} v$, is called the **closed star** of v. The set $\overline{\operatorname{St}} v \setminus \operatorname{St} v$ is called the **link** of v and is denoted by $\operatorname{Lk} v$.

In the following example, there are three 1-simplices and two 2-simplices that contain v_1 as a vertex. The interior of all these 5 simplices make $\operatorname{St} v_1$. $\operatorname{Lk} v_1$ is the L-shaped 1-dimensional subcomplex that is drawn in bold.



Since interior of a simplex is open, it follows immediately that $\operatorname{St} v$ is open; since union of open sets is also open. Note that, star of a vertex contains that vertex itself. Because the 0-simplex v is its own interior, so it is included in the union of the interior of all the simplices that have v as a vertex. That's why $\operatorname{Lk} v$ does not contain v, since v is in both $\operatorname{St} v$ and $\overline{\operatorname{St}} v$.

Furthermore, each point of $\operatorname{St} v$ belongs to the interior of a simplex that has v as vertex. Suppose $w \in \operatorname{St} v$, and $w \in \operatorname{Int} \sigma$ where v is a vertex of σ. Let l denote the line segment joining w and v. Then $l \setminus \{v\}$ lies in $\operatorname{Int} \sigma$. As a result, $l \subseteq \operatorname{St} v$. This illustrates that $\operatorname{St} v$ is a star-convex set with respect to the vertex v.

Theorem 2.1.1

Let K be a complex. Then the group $H_0(K)$ is free abelian. If $\{v_\alpha\}_\alpha$ is a collection consisting of one vertex from each component of $|K|$, then the homology classes of the chains v_α form a basis for $H_0(K)$.

Proof. Step 1. Let v and w be two vertices of K. Let us define a relation $v \sim w$ if there exists a sequence $a_0, a_1, \dots, a_n$ of vertices of K such that $v = a_0$, $w = a_n$, and $a_i a_{i+1}$ is a 1-simplex of K for each $i = 0, 1, \dots, n-1$. It's easy to check that $\sim$ is an equivalence relation.

Given a vertex v of K, we define the following set

$$C_v = \bigcup \{\operatorname{St} w \ : \ w \sim v\}$$

Our goal is to show that C_v are components of K. C_v is union of open sets, hence open. Now we shall show that each C_v is path connected.

Let's take any $x \in C_v$. We shall show that there is a path from v to x. This will suffice to show the path-connectedness of C_v. Since $x \in C_v$, $x \in \operatorname{St} w$ for some w with $w \sim v$.

Let $a_0 = v, a_1, \ldots, a_n = w$ be a sequence such that $a_i a_{i+1}$ is a 1-simplex of K for each $i = 0, 1, \ldots, n-1$. Since $x \in \operatorname{St} a_n$, the line segment $a_n x$ lies in $\operatorname{St} a_n$; because $\operatorname{St} a_n$ is star-convex with respect to a_n.

$a_i \sim v$ for each i, so $\operatorname{St} a_i \subseteq C_v$ and $\operatorname{St} a_{i+1} \subseteq C_v$ for each $i = 0, 1, \ldots, n-1$. Therefore, $a_i a_{i+1} \subseteq (\operatorname{St} a_i \cup \operatorname{St} a_{i+1}) \subseteq C_v$. As a result, the broken line with consecutive vertices $a_0 = v, a_1, \ldots, a_n, x$ lies in C_v. It is a continuous path from v to x. Therefore, C_v is path connected, and thus connected.

Now we shall show that the distinct sets C_v and $C_{v'}$ are disjoint. Suppose the contrary, and $x \in C_v \cap C_{v'}$. Then there exists a vertex w with $x \in \operatorname{St} w$ and $w \sim v$; also there exists another vertex w' with $x \in \operatorname{St} w'$ and $w' \sim v'$.

Now, x is contained in the interior of a simplex that has w as a vertex. Therefore, the barycentric coordinate $t_w(x)$ of x with respect to w is strictly positive. So, $t_w(x) > 0$. Similarly, $t_{w'}(x) > 0$.

But x lies in the interior of exactly one simplex of K. Since both $t_w(x)$ and $t_{w'}(x)$ are strictly positive, that simplex has both w and w' as vertices. This means ww' is a 1-simplex of K, so $w \sim w'$. Therefore, $v \sim w \sim w' \sim v'$, so $v \sim v'$ and hence $C_v = C_{v'}$. This leads to a contradiction, since we assumed C_v and $C_{v'}$ to be distinct. Hence, $C_v \cap C_{v'} = \varnothing$.

Now, C_v are open, disjoint, connected, and their union is the whole $|K|$. Therefore, they are necessarily the components of $|K|$. Note that, each such C_v is the underlying space of a subcomplex of K. Each simplex of K lies entirely in exactly one of the C_v's.

Step 2. Now we shall prove the theorem. Let $\{v_\alpha\}_\alpha$ be a collection of vertices containing one vertex v_α from each component C_α of $|K|$. Given a vertex w of K, it belongs to a component of K, say C_α.

In other words, $w \sim v_\alpha$. This means there exists a sequence $a_0 = v_\alpha, a_1, \ldots, a_n = w$ of vertices of K such that $a_i a_{i+1}$ is a 1-simplex of K for each $i = 0, 1, \ldots, n-1$.

The 1-chain given by $[a_0, a_1] + [a_1, a_2] + \cdots + [a_{n-1}, a_n]$ has boundary $a_n - a_0 = w - v_\alpha$. In other words,

$$w = v_\alpha + \partial_1 \left([a_0, a_1] + [a_1, a_2] + \cdots + [a_{n-1}, a_n]\right)$$

As a result, the 0-chain w is homologous to the 0-chain v_α. (Note the abuse of notation – we used the same symbol for the vertex and the elementary 0-chain corresponding to the vertex) Hence, every elementary 0-chain is homologous to one of the v_α's from the collection $\{v_\alpha\})_\alpha$. Therefore, every 0-chain of K is homologous to an integral linear combination of the v_α's, *i.e.* $c = \sum_\alpha n_\alpha v_\alpha$ with $n_\alpha \in \mathbb{Z}$.

Now we shall prove that no nontrivial chain of the form $c = \sum_\alpha n_\alpha v_\alpha$ bounds, which will establish that $B_0(K)$ is trivial. Assume the contrary that $c = \partial_1 d$ for some 1-chain d of K.

We have seen earlier in *step 1* that each simplex of K lies entirely in one and excatly one of the components of $|K|$. Therefore, each 1-simplex of K lies in a unique component of $|K|$. Now, using the exact same arguments as above, we get that

$$d = \sum_\alpha m_\alpha d_\alpha \ , \ m_\alpha \in \mathbb{Z} \, ,$$

and d_α's are elementary 1-chains carried by the components C_α's of $|K|$.

$$\sum_\alpha n_\alpha v_\alpha = c = \partial_1 d = \partial_1 \left(\sum_\alpha m_\alpha d_\alpha \right) = \sum_\alpha m_\alpha \ \partial_1 d_\alpha$$

Since d_α is carried by C_α, then so is $m_\alpha \partial_1 d_\alpha$. Therefore, $n_\alpha v_\alpha = m_\alpha \partial_1 d_\alpha$. Now we claim that $n_\alpha = 0$.

Let $\epsilon : C_0(K) \to \mathbb{Z}$ be a group homomorphism, such that $\epsilon(v) = 1$ for each elementary 0-chain v. Thus we have $\epsilon(n_\alpha v_\alpha) = n_\alpha$. Also,

$$\epsilon\left(\partial_1 [v, w]\right) = \epsilon(w - v) = \epsilon(w) - \epsilon(v) = 1 - 1 = 0$$

for any elementary 1-chain $[v, w]$. Therefore, $\epsilon(\partial_1 d_\alpha) = 0$. Thus we have

$$n_\alpha = \epsilon(n_\alpha v_\alpha) = \epsilon(m_\alpha \partial_1 d_\alpha) = m_\alpha \ \epsilon(\partial_1 d_\alpha) = 0$$

So $n_\alpha = 0$ for each α. Therefore, $\partial_1 d = c = 0$. In other words, a 0-chain that bounds is necessarily trivial. So $B_0(K)$ is trivial.

Since $c = \sum_\alpha n_\alpha v_\alpha$ for a given 0-chain (and hence a 0-cycle), and there are no non-trivial 0-chain that bounds, $\{v_\alpha\}_\alpha$ is a basis for $H_0(K)$. ■

§2.2 Reduced Homology

Definition 2.2.1 (Augmentation Map)**.** Let $\epsilon : C_0(K) \to \mathbb{Z}$ be the surjective homomorphism defined by $\epsilon(v) = 1$ for each vertex elementary 0-chain of K corresponding to the vertex v. Then if c is a 0-chain, $\epsilon(c)$ equals the sum of the values of c on the vertices of K. The map ϵ is called **an augmentation map** for $C_0(K)$.

We have seen in the proof of Theorem 2.1.1 that $\epsilon(\partial_1 d) = 0$ for a 1-chain d of K.

Definition 2.2.2 (Reduced Homology Group)**.** We define the **reduced homology group** of K in dimension 0, denoted $\widetilde{H}_0(K)$, by the equation

$$\widetilde{H}_0(K) = \operatorname{Ker}\epsilon / \operatorname{im}\partial_1$$

If $p > 0$, we let $\widetilde{H}_p(K) = H_p(K)$.

Theorem 2.2.1

The group $\widetilde{H}_0(K)$ is free abelian and $\widetilde{H}_0(K) \oplus \mathbb{Z} \cong H_0(K)$. Thus $\widetilde{H}_0(K)$ vanishes if $|K|$ is connected. If $|K|$ is not connected, let $\{v_\alpha\}_\alpha$ consist of one vertex from each component of $|K|$; also let α_0 be a fixed index. Then the homology classes of the chains $v_\alpha - v_{\alpha_0}$, for $\alpha \neq \alpha_0$, form a basis for $\widetilde{H}_0(K)$.

Proof. Given a 0-chain c on K, we've seen earlier that c is homologous to a 0-chain of the form $c' = \sum_\alpha n_\alpha v_\alpha$; and c' bounds only if $c' = 0$, *i.e.* $n_\alpha = 0$ for each α.

If $c \in \operatorname{Ker}\epsilon$, then $\epsilon(c) = 0$. Since c and c' are homologous, $c' = c + \partial_1 d$ for some 1-chain d. Therefore,

$$\epsilon(c') = \epsilon(c + \partial_1 d) = \epsilon(c) + \epsilon(\partial_1 d) = \epsilon(c)$$

$$\therefore 0 = \epsilon(c) = \epsilon(c') = \epsilon\left(\sum_\alpha n_\alpha v_\alpha\right) = \sum_\alpha n_\alpha \epsilon(v_\alpha) = \sum_\alpha n_\alpha$$

If $|K|$ is connected, there is only one component. And hence there is only one n_α. Thus $n_\alpha = 0$ gives us $c' = 0$. So $\operatorname{Ker}\epsilon$ is trivial. Since $\widetilde{H}_0(K) = \operatorname{Ker}\epsilon / \operatorname{im}\partial_1$ and $\operatorname{Ker}\epsilon$ is trivial, we get that $\widetilde{H}_0(K) = 0$.

Now suppose $|K|$ has more that one component. Let's fix an index α_0.

$$0 = \sum_\alpha n_\alpha = n_{\alpha_0} + \sum_{\alpha \neq \alpha_0} n_\alpha \implies n_{\alpha_0} = -\sum_{\alpha \neq \alpha_0} n_\alpha$$

Substituting this expression into $c' = \sum_\alpha n_\alpha v_\alpha$, we get

$$\begin{aligned} c' = \sum_\alpha n_\alpha v_\alpha &= \sum_{\alpha \neq \alpha_0} n_\alpha v_\alpha + n_{\alpha_0} v_{\alpha_0} \\ &= \sum_{\alpha \neq \alpha_0} n_\alpha v_\alpha + \left(-\sum_{\alpha \neq \alpha_0} n_\alpha\right) v_{\alpha_0} \\ &= \sum_{\alpha \neq \alpha_0} n_\alpha (v_\alpha - v_{\alpha_0}) \end{aligned}$$

Therefore, c' is a linear combination of the 0-chains $v_\alpha - v_{\alpha_0}$. And as before, such a 0-chain bounds only if it is trivial. Hence, homology classes[1] of the 0 chains $\{v_\alpha - v_{\alpha_0}\}_{\alpha \neq \alpha_0}$ forms a basis for $\widetilde{H}_0(K)$. ■

§2.3 The Homology of a Cone

Now our goal is to compute the homology groups of an n-simplex and of its boundary. For this, we shall develop this very useful tool called **cone**.

§2.3.i Generalized Euclidean Space $\mathbb{E}^J$

Let J be an arbitrary index set, and let $\mathbb{R}^j$ denote the J-fold product of $\mathbb{R}$ with itself. For instance, we can have $\mathbb{R}^{\mathbb{N}}$ $\mathbb{R}^{\mathbb{Q}}$ or $\mathbb{R}^{\mathbb{R}}$. An element of $\mathbb{R}^J$ is treated as a function from J to $\mathbb{R}$. One usually uses the follwoing "tuple" notation for an element of $\mathbb{R}^J$:

$$\mathbb{R}^J \ni x = (x_\alpha)_{\alpha \in J}$$

$\mathbb{R}^J$ is, of course, a vector space over the field $\mathbb{R}$ with the usual component-wise addition and scalar multiplication.

$$(x_\alpha)_{\alpha \in J} + (y_\alpha)_{\alpha \in J} = (x_\alpha + y_\alpha)_{\alpha \in J} \quad \text{and} \quad k \cdot (x_\alpha)_{\alpha \in J} = (k \cdot x_\alpha)_{\alpha \in J}$$

Let $\mathbb{E}^J$ denote the subset of $\mathbb{R}^J$ consisting of all points $(x_\alpha)_{\alpha \in J}$ such that $x_\alpha = 0$ for all but finitely many $\alpha \in J$. In other words, if $(x_\alpha)_{\alpha \in J} \in \mathbb{E}^J$ then there can be only finitely many nonzero coordinate appearing in this J-tuple.

We define a map $\varepsilon_\alpha : J \to \mathbb{R}$ as follows:

$$\varepsilon_\alpha(x) = \begin{cases} 1 & \text{if } x = \alpha \\ 0 & \text{if } x \neq \alpha \end{cases}$$

Then the set $\{\varepsilon_\alpha \ : \ \alpha \in J\}$ is a basis for $\mathbb{E}^J$, but not a basis for $\mathbb{R}^J$ (left as an exercise for the reader).

We call $\mathbb{E}^J$ **generalized Euclidean space** and topologize it by introducing the following metric:

$$x \equiv (x_\alpha)_{\alpha \in J} \ , \ y \equiv (y_\alpha)_{\alpha \in J} \ ; \ d(x, y) = \max\{|x_\alpha - y_\alpha \ : \ \alpha \in J|\}$$

Everything that we've done for complexes in $\mathbb{R}^n$ ($n \in \mathbb{N}$) can be extended for complexes in $\mathbb{E}^J$.

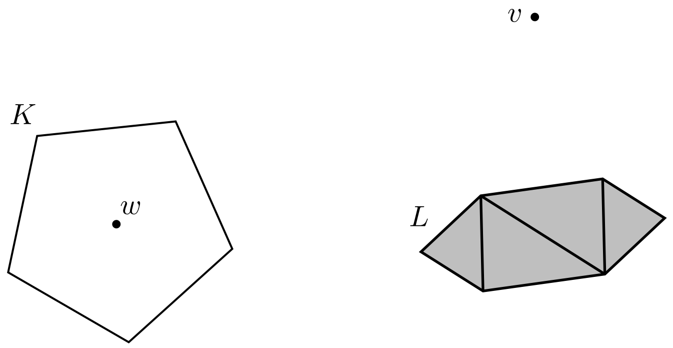
§2.3.ii Cone

> **Definition 2.3.1** (Cone). Let K be a complex in $\mathbb{E}^J$ and $w \in \mathbb{E}^J$ such that every ray emanating from from w intersects K in at most one point. Then we define the **cone** on K with vertex w to be the collection of all simplices of the form $[w, a_0, \ldots, a_p]$ ($[a_0, \ldots, a_p]$ is a simplex of K) along with all the faces of such simplices. We denote this collection by $w * K$.

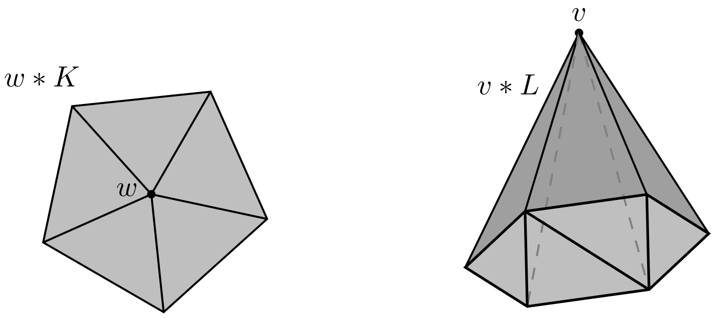
It is left as an exercise for the reader to verify that $w * K$ is indeed a well-defined complex, and it contains K as a subcomplex. K is called the base of the cone.

In the following figure, K and L are respectively a 1-dimensional complex in $\mathbb{R}^2$ and a 2-dimensional complex in RR^3. $w \in \mathbb{R}^2$ and $v \in \mathbb{R}^3$ such that every ray emanating from from w intersects K in at most one point; and every ray emanating from from v intersects K in at most one point.

[1] Two 0-chains $v_\alpha - v_{\alpha_0}$ and $v_{\alpha'} - v_{\alpha_0}$ are equivalent, or they belong in the same equivalence class iff $v_\alpha - v_{\alpha_0} = v_{\alpha'} - v_{\alpha_0} + \partial_1 d$ for some 1-chain d.

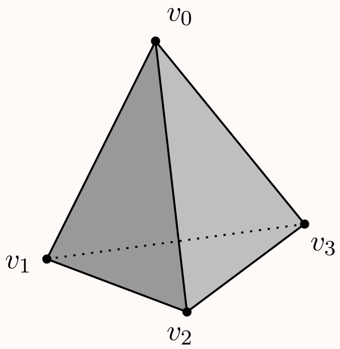


The cones $w * K$ and $v * L$ are shown below:



Example 2.3.1

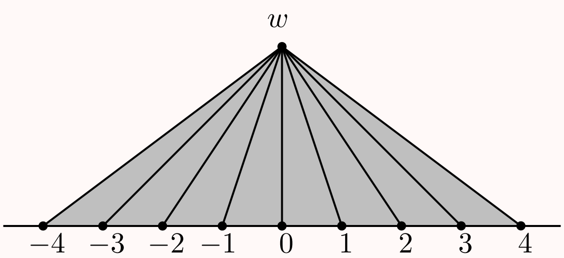
Let K_σ denote the complex consisting of the n-simplex $\sigma = [v_0, v_1, \ldots, v_n]$ and all its faces. Then $K_\sigma = v_0 * K_s$, where s is the face of σ opposite to v_0.



For instance, in the figure above, $\sigma = [v_0, v_1, v_2, v_3]$, $s = [v_1, v_2, v_3]$; and $K_\sigma = v_0 * K_s$. Therefore, every simplex in positive dimension is a cone.

Example 2.3.2

Let K be the complex in $\mathbb{R}^2$ consisting of the intervals $[n, n+1] \times 0$ for $n \in \mathbb{Z}$. Then $|K|$ is the x-axis. Let w be a point on the y-axis different from the origin. Then $w * K$ is illustrated below:

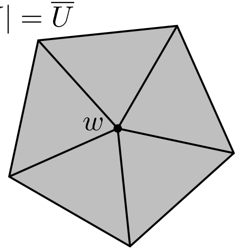


Although $|K|$ is a subspace of $\mathbb{R}^2$, $|w * K|$ is not a subspace of $\mathbb{R}^2$. Because according to *§2,* *Exercise 9* of Munkres's book, "$|K|$ is a subspace of $\mathbb{R}^N$ if and only if each point x of $|K|$ lies in an open set of $\mathbb{R}^N$ that intersects only finitely many simplices of K."

Lemma 2.3.1

Let U be a bounded convex open set in $\mathbb{R}^n$; let $w \in U$. If K is a finite complex such that $|K| = \overline{U} \setminus U$, then $w * K$ is a finite complex such that $|w * K| = \overline{U}$.

Proof. According to *Lemma 1.1* of of Munkres's book, each ray emanating from w intersects $|K| = \overline{U} \setminus U$ in exactly one point.



Therefore, the cone $w * K$ is well-defined. $\overline{U}$ is the union of all line segments joining w to points of $|K|$. And hence, $\overline{U} = |w * K|$. ■

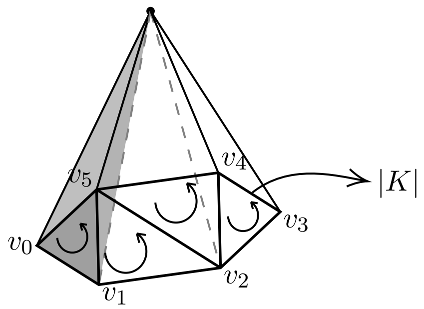
Definition 2.3.2. Let $w * K$ be a cone. If $\sigma = [a_0, \ldots, a_p]$ is an oriented simplex of K, let $[w, \sigma]$ denote the oriented simplex $[w, a_0, \ldots, a_p]$ of $w * K$.

This bracket operation $[w, \sigma]$, where σ is an oriented simplex of K, is well-defined. If $c_p = \sum n_i \sigma_i$ is a p-chain of K, we define

$$[w, c_p] = \sum n_i [w, \sigma_i] .$$

This **bracket-operation** is a group homomorphism from $C_p(K)$ to $C_{p+1}(w * K)$, *i.e.* from the group of p-chains on K to the group of $(p+1)$-chains on the cone $w * K$.

For example, in the following figure, K is the complex consisting of $[v_0, v_1, v_5]$, $[v_5, v_1, v_4]$, $[v_4, v_1, v_2]$, $[v_2, v_3, v_4]$ and their faces.



If $\sigma = [v_0, v_1, v_5]$, then $[w, \sigma]$ is the 3-simplex $[w, v_0, v_1, v_5]$ which is colored in gray.

Proposition 2.3.2

Let σ be an elementary p-chain. Then

$$\partial_{p+1} [w, \sigma] = \begin{cases} \sigma - w & \text{if } p = 0 \\ \sigma - [w, \partial_p \sigma] & \text{if } p > 0 \end{cases}$$

Proof. The first one is obvious : if σ is an elementary 0-chain, then $[w, \sigma]$ is an elementary 1-chain, which yields $\partial_1 [w, \sigma] = \sigma - w$.

Now let's suppose $p > 0$; and $\sigma = [v_0, v_1, \dots, v_p]$. Then we know that

$$\partial_p \sigma = \partial_p [v_0, v_1, \dots, v_p] = \sum_{i=0}^{p} (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_p]$$

Now $[w, \sigma] = [w, v_0, v_1, \dots, v_p]$ is an elementary $(p+1)$-chain. Let's compute $\partial_{p+1} [w, \sigma]$.

$$\begin{aligned}\partial_{p+1} [w, \sigma] &= \partial_{p+1} [w, v_0, \dots, v_p] \\ &= [v_0, \dots, v_p] + \sum_{i=0}^{p} (-1)^{i+1} [w, v_0, \dots, \widehat{v_i}, \dots, v_p] \\ &= \sigma + \left[w, \sum_{i=0}^{p} (-1)^{i+1} [v_0, \dots, \widehat{v_i}, \dots, v_p] \right] = \sigma - [w, \partial_p \sigma]\end{aligned}$$

■

Proposition 2.3.3

Let c_p be a p-chain. Then

$$\partial_{p+1} [w, c_p] = \begin{cases} c_p - \epsilon(c_p)\ w & \text{if } p = 0 \\ c_p - [w, \partial_p c_p] & \text{if } p > 0 \end{cases}$$

Proof. Let $c_0 = \sum n_i \sigma_i^0$ be a 0-chain, where σ_i^0 are elementary 0-chains. Then

$$\epsilon(c_0) = \epsilon\left(\sum_i n_i \sigma_i^0\right) = \sum_i n_i\ \epsilon(\sigma_i^0) = \sum_i n_i$$

Now let's compute $\partial_1 [w, c_0]$.

$$\begin{aligned}\partial_1 [w, c_0] &= \partial_1 \left(\left[w, \sum_i n_i \sigma_i^0 \right] \right) = \partial_1 \left(\sum_i n_i [w, \sigma_i^0] \right) \\ &= \sum_i n_i \partial_1 ([w, \sigma_i^0]) = \sum_i n_i (\sigma_i^0 - w) \\ &= \sum_i n_i \sigma_i^0 - \sum_i n_i\ w = c_0 - \epsilon(c_0)\ w\end{aligned}$$

Now let $c_p = \sum n_i \sigma_i^p$ be a p-chain on the base of the cone, where σ_i^p are elementary p-chains. Then $[w, c_p]$ is a $(p+1)$-chain on the cone.

$$\begin{aligned}\partial_{p+1} [w, c_p] &= \partial_{p+1} \left(\left[w, \sum_i n_i \sigma_i^p \right] \right) = \partial_{p+1} \left(\sum_i n_i [w, \sigma_i^p] \right) \\ &= \sum_i n_i\ \partial_{p+1} [w, \sigma_i^p] = \sum_i n_i (\sigma_i^p - [w, \partial_p \sigma_i^p]) \\ &= \sum_i n_i \sigma_i^p - \sum_i n_i [w, \partial_p \sigma_i^p] = c_p - \left[w, \partial_p \left(\sum_i n_i \sigma_i^p \right) \right] \\ &= c_p - [w, \partial_p c_p]\end{aligned}$$

■

> **Definition 2.3.3** (Acyclic). A complex whose reduced homology groups vanishes in all dimensions is said to be **acyclic**.

> **Theorem 2.3.4** (Reduced Homology Groups of Cone)
> If $w * K$ is a cone, then
> $$\widetilde{H}_p(w * K) = 0 \qquad \forall p\ .$$
> In other words, $w * K$ is acyclic.

Proof. Firstly, notice that the underlying space $|w * K|$ of the cone is connected. In fact, $|w * K|$ is path connected. Because if we take a point x from $|w * K|$, there is a unique simplex σ of $w * K$ such that $x \in \operatorname{Int}\sigma$. If σ is not carried by K, then w is a vertex of σ, and hence the line segment wx is in $|w * K|$. Otherwise, if σ is carried by K, the simplex $[w, \sigma]$ contains the line segment wx.

Thus, $|w * K|$ is path connected, and hence connected. Since $|w * K|$ is connected, by Theorem 2.2.1, $\widetilde{H}_0(w * K) = 0$.

Now let's consider the case $p > 0$. Let $z_p \in Z_p(w * K)$ be a p-cycle of $w * K$. We want to show that z_p bounds. Let's decompose z_p as follows:

$$z_p = c_p + [w, d_{p-1}]\ ,$$

where c_p consists of the terms carried by K, and d_{p-1} is a $(p-1)$-chain on K. We claim that $z_p = \partial_{p+1}[w, c_p]$.

$$\begin{aligned} z_p - \partial_{p+1}[w, c_p] &= z_p - c_p + [w, \partial_p c_p] = [w, d_{p-1}] + [w, \partial_p c_p] \\ &= [w, d_{p-1} + \partial_p c_p] = [w, e_{p-1}] \end{aligned}$$

where $e_{p-1} = d_{p-1} + \partial_p c_p$ is a $(p-1)$-chain on K. Since z_p is a p-cycle, $\partial_p z_p = 0$.

$$\begin{aligned} & z_p - \partial_{p+1}[w, c_p] = [w, e_{p-1}] \\ \implies & \partial_p [w, e_{p-1}] = \partial_p z_p - \partial_p \partial_{p+1}[w, c_p] = 0 \\ \implies & 0 = \partial_p [w, e_{p-1}] = \begin{cases} e_{p-1} - \epsilon(e_{p-1})\ w & \text{if } p = 1 \\ e_{p-1} - [w, \partial_{p-1} e_{p-1}] & \text{if } p > 1 \end{cases} \\ \implies & e_{p-1} = \begin{cases} \epsilon(e_{p-1})\ w & \text{if } p = 1 \\ [w, \partial_{p-1} e_{p-1}] & \text{if } p > 1 \end{cases} \end{aligned}$$

We know that $e_{p-1} = d_{p-1} + \partial_p c_p$ is a $(p-1)$-chain carried by K. But the RHS of this equation suggests that e_{p-1} admits contribution from $(w * K) \setminus K$ too. This can be true only when e_{p-1} is trivial, *i.e.* $e_{p-1} = 0$.

$$z_p - \partial_{p+1}[w, c_p] = [w, e_{p-1}] = [w, 0] = 0 \implies z_p = \partial_{p+1}[w, c_p]$$

Therefore, $z_p \in B_p(w * K)$. As a result, $Z_p(w * K) = B_p(w * K)$. Hence,

$$\widetilde{H}_p(w * K) = H_p(w * K) = Z_p(w * K) / B_p(w * K) = 0\,.$$

■

> **Theorem 2.3.5**
> Let σ be an n-simplex. The complex K_σ consisting of σ and its faces is acyclic. If $n > 0$, let Σ^{n-1} denote the complex whose polytope is $\operatorname{Bd}\sigma$. Orient σ. Then $\widetilde{H_{n-1}}\left(\Sigma^{n-1}\right)$ is infinite cyclic and is generated by the chain $\partial_n \sigma$; furthermore, $\widetilde{H}_i\left(\Sigma^{n-1}\right) = 0$ for $i \neq n-1$.

Proof. By Example 2.3.1, K_σ is a cone. Then by Theorem 2.3.4, K_σ is acyclic. Let us compare the chain groups of K_σ and Σ^{n-1}; they are equal except in dimension n.

$$\begin{array}{ccccccccccc} 0 & \xrightarrow{\partial_{n+1}} & C_n(K_\sigma) & \xrightarrow{\partial_n} & C_{n-1}(K_\sigma) & \xrightarrow{\partial_{n-1}} & \cdots & \xrightarrow{\partial_1} & C_0(K_\sigma) & \xrightarrow{\epsilon} & \mathbb{Z} \\ & & & & \| & & & & \| & & \| \\ & & 0 & \xrightarrow{\partial'_n} & C_{n-1}(\Sigma^{n-1}) & \xrightarrow{\partial'_{n-1}} & \cdots & \xrightarrow{\partial'_1} & C_0(\Sigma^{n-1}) & \xrightarrow{\epsilon'} & \mathbb{Z} \end{array}$$

For $n > 0$, $C_{n-1}(K_\sigma) = C_{n-1}(\Sigma^{n-1})$. Recall that the $(n-1)$-chains on a given complex are maps from $(n-1)$-simplices of the complex to $\mathbb{Z}$. Since $\left|\Sigma^{n-1}\right| = \operatorname{Bd}\sigma$, the $(n-1)$-simplices of K_σ coincide with the $(n-1)$-simplices of Σ^{n-1}. Hence, the $(n-1)$-chains of the respective complexes also coincide.

Thus, we immediately see that

$$H_i(K_\sigma) = H_i(\Sigma^{n-1}) \ , \text{ for } i \neq n-1 \implies 0 = \widetilde{H_i}(K_\sigma) = \widetilde{H_i}(\Sigma^{n-1}) \ , \text{ for } i \neq n-1$$

Let's now compute the homology group in dimension $n-1$. From the diagram above, we can see that $\operatorname{im}\partial'_n$ is trivial, because it maps a trivial group to $C_{n-1}(\Sigma^{n-1})$. Also, $\partial_m = \partial'_m$ for $m \leq n-1$.

Since K_σ is acyclic, $\widetilde{H_i}(K_\sigma) = 0$. Therefore, for $n > 1$,

$$0 = \widetilde{H_{n-1}}(K_\sigma) = H_{n-1}(K_\sigma) = \operatorname{Ker}\partial_{n-1}/\operatorname{im}\partial_n \implies \operatorname{Ker}\partial_{n-1} = \operatorname{im}\partial_n$$

Using these facts, we can now compute $H_{n-1}(\Sigma^{n-1})$.

$$\begin{aligned} H_{n-1}(\Sigma^{n-1}) &= Z_{n-1}(\Sigma^{n-1})/B_{n-1}(\Sigma^{n-1}) = \operatorname{Ker}\partial'_{n-1}/\operatorname{im}\partial'_n \\ &= \operatorname{Ker}\partial'_{n-1} = \operatorname{Ker}\partial_{n-1} = \operatorname{im}\partial_n \end{aligned}$$

We know that $C_n(K_\sigma)$ is free abelian, with basis being the set of elementary n-chains. Here we have only one elementary n-chain, corresponding to the oriented n-simplex σ. Therefore, $C_n(K_\sigma) \cong \mathbb{Z}$ with σ being the generator of $C_n(K_\sigma)$.

We've just seen that $C_n(K_\sigma)$ is an infinite cyclic group. $\partial_n : C_n(K_\sigma) \to C_{n-1}(K_\sigma)$ is a group homomorphism; and homomorphic image of cyclic group is also cyclic. Therefore, $\operatorname{im}\partial_n$ is cyclic with generator $\partial_n\sigma$.

$C_{n-1}(K_\sigma)$ is free, and hence torsion free. $\operatorname{im}\partial_n$ is a cyclic subgroup of this torsion free group. No element of $C_{n-1}(K_\sigma)$ has a finite order since $C_{n-1}(K_\sigma)$ is torsion free. Therefore, the order of $\partial_n\sigma$ is not finite. As a result, $\operatorname{im}\partial_n$ is a infinite cyclic group with generator $\partial_n\sigma$. Therefore, for $n > 1$,

$$\widetilde{H_{n-1}}(\Sigma^{n-1}) = H_{n-1}(\Sigma^{n-1}) = \operatorname{im}\partial_n \cong \mathbb{Z} \ ,$$

and $\widetilde{H_{n-1}}(\Sigma^{n-1})$ is generated by the $(n-1)$-chain $\partial_n\sigma$.

The proof for $n = 1$ case is exactly similar, with ∂_{n-1} replaced by ϵ. ■

3 Lecture 3

§3.1 Relative Homology

Suppose K_0 is a subcomplex of a given complex K. Then $C_p(K_0)$ can be seen to be a subgroup of $C_p(K)$. Any group element in $C_p(K_0)$ can be seen to be a group element of $C_p(K)$ in the following way: If c_p is a p-chain on K_0, one extends it to be a p-chain on K by letting its value be 0 on each oriented p-simplex on K that is not in K_0. Now, since $C_p(K_0)$ is a subgroup of $C_p(K)$, it makes sense to talk about the quotient group $C_p(K)/C_p(K_0)$.

> **Definition 3.1.1** (Group of Relative Chains)**.** If K_0 is a subcomplex of K, the quotient group $C_p(K)/C_p(K_0)$ is called **the group of relative chains** of K modulo K_0. It is generally denoted by $C_p(K, K_0)$.

The group $C_p(K, K_0)$ is free, for it has basis all cosets of the form

$$\{\sigma_i\} = \sigma_i + C_p(K_0) \ ,$$

where σ_i is an elementary p-chain corresponding to the oriented p-simplex σ_i of K that is not in K_0.

Given the boundary operator $\partial_p : C_p(K) \to C_{p-1}(K)$, its restriction on $C_p(K_0)$ is denoted by the same symbol; $\partial_p : C_p(K_0) \to C_{p-1}(K_0)$. This homomorphism induces a homomorphism (which we denote by the same symbol ∂_p) of the relative chain groups.

$$\partial_p : C_p(K, K_0) \to C_{p-1}(K, K_0) \ ; \qquad \sigma_i + C_p(K_0) \mapsto \partial_p\sigma_i + C_{p-1}(K_0) \ .$$

$$C_{p+1}(K, K_0) \xrightarrow{\partial_{p+1}} C_p(K, K_0) \xrightarrow{\partial_p} C_{p-1}(K, K_0)$$

As before, this boundary operator ∂_p satisfies $\partial_p \circ \partial_{p+1} = 0$.

> **Definition 3.1.2** (Relative Homology Group)**.** The **group of relative p-cycles** of K modulo K_0, denoted by $Z_p(K, K_0)$, is defined as
>
> $$Z_p(K, K_0) = \operatorname{Ker} \partial_p \ , \qquad \text{where } \partial_p : C_p(K, K_0) \to C_{p-1}(K, K_0) \ .$$
>
> Similarly, the **group of relative p-boundaries** of K modulo K_0, denoted by $B_p(K, K_0)$, is defined as
>
> $$B_p(K, K_0) = \operatorname{im} \partial_{p+1} \ , \qquad \text{where } \partial_{p+1} : C_{p+1}(K, K_0) \to C_p(K, K_0) \ .$$
>
> The **relative homology group** of K modulo K_0 in dimension p, denoted by $H_p(K, K_0)$, is defined as
>
> $$H_p(K, K_0) = Z_p(K, K_0)/B_p(K, K_0) \ .$$

Remark. A relative p-chain $c_p + C_p(K_0) \in C_p(K, K_0)$ is a relative p-cycle, *i.e.* $c_p + C_p(K_0) \in Z_p(K, K_0)$, if and only if $\partial_p c_p$ is carried by K_0. Furthermore, $c_p + C_p(K_0) \in B_p(K, K_0)$ if and only if there exists a $(p+1)$-chain d_{p+1} on K such that $c_p - \partial_{p+1}d_{p+1}$ is carried by K_0.

Example 3.1.1. Let K consist of an n-simplex and its faces; and let K_0 be the set of proper faces of K_0. Now, consider the group $C_p(K, K_0) = C_p(K)/C_p(K_0)$. If $p > n$, then $C_p(K)$ is trivial, thus $C_p(K, K_0)$ is trivial. If $p < n$, then $C_p(K)$ coincides with $C_p(K_0)$, so $C_p(K, K_0)$ is trivial. In other words, $C_p(K, K_0) = 0$ unless $p = n$.

When $p = n$, $C_p(K_0)$ is trivial, because there is no n-simplex in K_0. So $C_n(K, K_0) = C_n(K) \cong \mathbb{Z}$. ($C_n(K)$ is free abelian; in fact, it is infinite cyclic, generated by the elementary n-chain σ corresponding to the n-simplex. So $C_n(K) \cong \mathbb{Z}$)

Now, since $C_p(K, K_0) = 0$ for every $p \neq 0$, we have

$$0 \xrightarrow{\partial_{n+1}} C_n(K, K_0) \xrightarrow{\partial_n} 0$$

Therefore, $\operatorname{Ker} \partial_n = C_n(K, K_0)$ and $\operatorname{im} \partial_{n+1} = 0$. As a result,

$$H_n(K, K_0) = \operatorname{Ker} \partial_n / \operatorname{im} \partial_{n+1} = C_n(K, K_0) \cong \mathbb{Z}$$

Since $C_p(K, K_0) = 0$ for $p \neq n$, $H_p(K, K_0) = 0$ for $p \neq n$. To summarize

$$H_p(K, K_0) \cong \begin{cases} 0 & \text{if } p \neq n \\ \mathbb{Z} & \text{if } p = n \end{cases}$$

Example 3.1.2. Let K be a complex and v_{α_0} be a vertex of K. Firstly, we are interested in $H_0(K, v_{\alpha_0})$. Since $C_0(K, v_{\alpha_0}) = C_0(K) / C_0(v_\alpha)$, an element in $C_0(K, v_{\alpha_0})$ can be expressed as

$$\sum_{\alpha \neq \alpha_0} n_\alpha v_\alpha$$

where $\{v_\alpha\}_{\alpha \neq \alpha_0}$ is the collection containing a vertex from each component of $|K|$ except the one that contains v_{α_0}. Any relative 0-chain of K modulo v_{α_0} is a relative 0-cycle of K modulo v_{α_0}. Using the argument presented in *Step 2* of Theorem 2.1.1, one easily finds that the group of relative p-boundaries of K modulo v_{α_0} is trivial. Hence,

$$H_0(K, v_{\alpha_0}) = Z_0(K, v_{\alpha_0}) = C_0(K, v_{\alpha_0})$$

$C_0(K, v_{\alpha_0})$ is free abelian and has $\{v_\alpha\}_{\alpha \neq \alpha_0}$ as a basis.

From Theorem 2.2.1, we knoe that the collection $\{v_\alpha - v_{\alpha_0}\}_{\alpha \neq \alpha_0}$ forms a basis for the 0-th reduced homology group $\widetilde{H_0}(K)$. Each basis element in $\{v_\alpha\}_{\alpha \neq \alpha_0}$ is in 1-1 correspondence with a basis element in $\{v_\alpha - v_{\alpha_0}\}_{\alpha \neq \alpha_0}$. Therefore, $H_0(K, v_{\alpha_0})$ and $\widetilde{H_0}(K)$ have the same rank, and hence they are isomorphic. $\boxed{H_0(K, v_{\alpha_0}) \cong \widetilde{H_0}(K)}$.

Let us now see what happens in dimension $p > 0$. $H_p(K, v_{\alpha_0}) = Z_p(K, v_{\alpha_0}) / B_p(K, v_{\alpha_0})$. Let $b_p \in B_p(K, v_{\alpha_0})$. Then b_p is a p-chain on K and there exists a $(p+1)$-chain d_{p+1} on K such that $b_p - \partial_{p+1} d_{p+1}$ is carried by v_{α_0}.

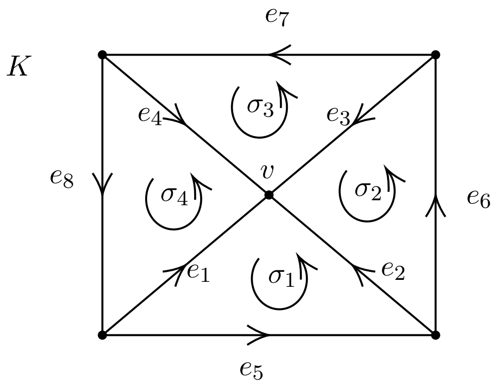
Here $p \geq 1$, and $b_p - \partial_{p+1} d_{p+1}$ is a p-chain. There is no nontrivial p-chain that is carried by v_{α_0}. Therefore, $b_p - \partial_{p+1} d_{p+1} = 0$. So $b_p \in B_p(K)$. Hence, $B_p(K, v_{\alpha_0}) \subseteq B_p(K)$.

Now, let $b_p \in B_p(K)$. Then b_p is a p-chain on K and there exists a $(p+1)$-chain d_{p+1} on K such that $b_p = \partial_{p+1} d_{p+1}$. So $b_p - \partial_{p+1} d_{p+1}$ is trivially carried by v_{α_0}. As a result, $b_p \in B_p(K, v_{\alpha_0})$, and hence $B_p(K, v_{\alpha_0}) \supseteq B_p(K)$. So we get $B_p(K, v_{\alpha_0}) = B_p(K)$.

Now let $c_p \in Z_p(K, v_{\alpha_0})$. Then $\partial_p c_p$ is carried by v_{α_0}. There is no nontrivial $(p-1)$-chain of the form $\partial_p c_p$ that is carried by v_{α_0} (convince yourself that this holds for $p = 1$ case too). Hence, $\partial_p c_p = 0$, so $c_p \in Z_p(K)$. Thus, $Z_p(K, v_{\alpha_0}) \subseteq Z_p(K)$. It's trivial to show that $Z_p(K, v_{\alpha_0}) \supseteq Z_p(K)$. So $Z_p(K, v_{\alpha_0}) = Z_p(K)$. Thus we have, for $p > 0$

$$H_p(K, v_{\alpha_0}) = Z_p(K, v_{\alpha_0}) / B_p(K, v_{\alpha_0}) = Z_p(K) / B_p(K) = H_p(K)$$

Example 3.1.3. Let K be the following complex whose underlying space is a square; and let K_0 be the subcomplex of K whose undelying space is the boundary of the square.



Now let $\sigma = \sum_{i=1}^{4} m_i \sigma_i \in Z_2(K, K_0)$. Then $\partial_2 \sigma$ is carried by K_0.

$$\begin{aligned}\partial_2 \sigma &= \sum_{i=1}^{4} m_i\, \partial_2 \sigma_i = m_1(e_5 + e_2 - e_1) + m_2(e_6 + e_3 - e_2) + m_3(e_7 + e_4 - e_3) + m_4(e_8 + e_1 - e_4) \\ &= m_1 e_5 + m_2 e_6 + m_3 e_7 + m_4 e_8 + (m_4 - m_1)\, e_1 + (m_1 - m_2)\, e_2 + (m_2 - m_3)\, e_3 + (m_3 - m_4)\, e_4\end{aligned}$$

Since $\partial_2 \sigma$ is carried by K_0, we must have $m_1 = m_2 = m_3 = m_4$. Therefore, $Z_2(K, K_0) \cong \mathbb{Z}$. Since there are no 3-simplices, $B_2(K, K_0)$ is trivial. Therefore,

$$H_2(K, K_0) = Z_2(K, K_0) / B_2(K, K_0) = Z_2(K, K_0) \cong \mathbb{Z}$$

We showed in the course *Algebraic Topology I* that any 1-chain c of the complex K is homologous to a 1-chain c_3 carried by $K_0 \cup e_4$. Now, if c is a relative 1-cycle of K modulo K_0, then $\partial_1 c$ is carried by K_0. Hence $\partial_1 c_3$ is also carried by K_0. Because

$$c_3 = c + \partial_2(a\sigma_1 + b\sigma_2 + c\sigma_2) \implies \partial_1 c_3 = \partial_1 c \quad (\text{because } \partial_1 \circ \partial_2 = 0)$$

If c_3 had nontrivial value on e_4, then $\partial_1 c_3$ would have nontrivial value on the vertex v, which is not the case since $\partial_1 c_3$ is carried by K_0. Therefore, c_3 is carried by K_0.

For any $c + C_1(K_0) \in Z_1(K, K_0)$, we found that it is homologous to $c_3 + C_1(K_0)$. Since c_3 is carried by K_0, we can conclude that $c_3 + C_1(K_0)$ is the trivial coset. Therefore, $Z_1(K, K_0)$ is trivial. As a result,

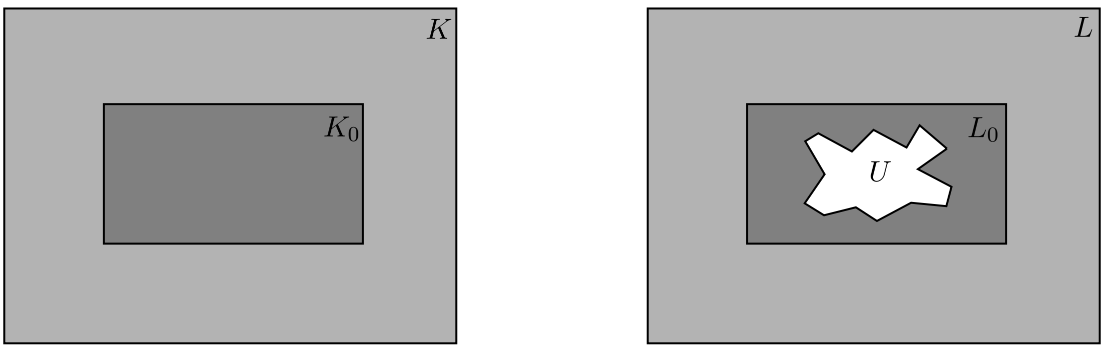
$$H_1(K, K_0) = Z_1(K, K_0) / B_1(K, K_0) = 0$$

Theorem 3.1.1 (Excision Theorem)

Let K be a complex and K_0 be a subcomplex of K. Let U be an open set contained in K_0, such that $|K| \setminus U$ is the polytope of a subcomplex L of K. Let L_0 be the subcomplex of K whose polytope is $|K_0| \setminus U$. Then the inclusion $L_0 \hookrightarrow L$ and $K_0 \hookrightarrow K$ induces an isomorphism

$$H_p(L, L_0) \cong H_p(K, K_0)$$

We think of the pair $(|L|, |L_0|)$ as having been formed by excising away the open set U from $|K|$ and $|K_0|$, respectively. See the figure below:



Proof. An element of $C_p(L)$ can be regarded as an element of $C_p(K)$ naturally by taking its value to be 0 on all p-simplices lying in $K \setminus L$. So one can form a composition map φ_p by taking the natural inclusion map $C_p(L) \hookrightarrow C_p(K)$ followed by the quotient map $\pi : C_p(K) \to C_p(K)/C_p(K_0)$.

$$C_p(L) \xrightarrow{\iota} C_p(K) \xrightarrow{\pi} C_p(K)/C_p(K_0), \qquad \varphi_p = \pi \circ \iota : C_p(L) \to C_p(K)/C_p(K_0)$$

So, if $c \in C_p(L)$, then $\iota(c) = c$ and thus $\varphi_p(c) = \pi(c) = c + C_p(K_0)$. Next we will show that φ_p is surjective.

Remember that $C_p(K)/C_p(K_0)$ has as basis all cosets $\{\sigma_i\}$, where σ_i are elementary p-chains on K carried by $K/K_0 \subseteq L$.

$$\varphi_p(\sigma_i) = \sigma_i + C_p(K_0)$$

Since every basis elements of $C_p(K)/C_p(K_0)$ has a preimage, each element of $C_p(K)/C_p(K_0)$ must have a preimage. Therefore, φ_p is surjective.

As a result, by the *First Isomorphism Theorem*,

$$C_p(L)/\operatorname{Ker}\varphi_p \cong C_p(K)/C_p(K_0)$$

with the isomorphism

$$c + \operatorname{Ker}\varphi_p \mapsto \varphi_p(c) = c + C_p(K_0)\ .$$

Now, our claim is that $\operatorname{Ker}\varphi_p = C_p(L_0)$. $L_0 \subseteq K_0$, so $C_p(L_0) \subseteq C_p(K_0)$. If $c_1 \in C_p(L_0)$, then

$$\varphi_p(c_1) = c_1 + C_p(K_0) = 0 + C_p(K_0) \implies c_1 \in \operatorname{Ker}\varphi_p$$

Hence, $C_p(L_0) \subseteq \operatorname{Ker}\varphi_p$.

Now suppose $c_2 \in \operatorname{Ker}\varphi_p$. Then

$$\varphi_p(c_2) = c_2 + C_p(K_0) = 0 + C_p(K_0)$$

Thus c_2 is a p-chain on K_0. Therefore, c_2 is a p-chain on $K_0 \cap L = L_0$. So $c_2 \in C_p(L_0)$. As a result, $C_p(L_0) \supseteq \operatorname{Ker}\varphi_p$.

Therefore, $C_p(L_0) = \operatorname{Ker}\varphi_p$. As a result,

$$C_p(L)/C_p(L_0) \cong C_p(K)/C_p(K_0)\ ,\quad \text{isomorphism } c + C_p(L_0) \mapsto c + C_p(K_0)\ .$$

We denote this isomorphism by $i_p : C_p(L, L_0) \to C_p(K, K_0)$.

$$\begin{array}{ccccc}
C_{p+1}(L, L_0) & \xrightarrow{\partial_{p+1}^{(L,L_0)}} & C_p(L, L_0) & \xrightarrow{\partial_p^{(L,L_0)}} & C_{p-1}(L, L_0) \\
\downarrow{\scriptstyle i_{p+1}} & & \downarrow{\scriptstyle i_p} & & \downarrow{\scriptstyle i_{p-1}} \\
C_{p+1}(K, K_0) & \xrightarrow{\partial_{p+1}^{(K,K_0)}} & C_p(K, K_0) & \xrightarrow{\partial_p^{(K,K_0)}} & C_{p-1}(K, K_0)
\end{array}$$

We are gonna show that the two squares of this diagram commutes. It suffices to show that $i_{p-1} \circ \partial_p^{(L,L_0)} = \partial_p^{(K,K_0)} \circ i_p$.

Take any $c + C_p(L_0) \in C_p(L, L_0)$. Then we have

$$\partial_p^{(L,L_0)}(c + C_p(L_0)) = \partial_p c + C_{p-1}(L_0)$$

$$\therefore \left(i_{p-1} \circ \partial_p^{(L,L_0)}\right)(c + C_p(L_0)) = i_{p-1}(\partial_p c + C_{p-1}(L_0)) = \boxed{\partial_p c + C_{p-1}(K_0)}$$

On the other hand,

$$i_p(c + C_p(L_0)) = c + C_p(K_0)$$

$$\therefore \left(\partial_p^{(K,K_0)} \circ i_p\right)(c + C_p(L_0)) = \partial_p^{(K,K_0)}(c + C_p(K_0)) = \boxed{\partial_p c + C_{p-1}(K_0)}$$

Therefore, we have

$$i_{p-1} \circ \partial_p^{(L,L_0)} = \partial_p^{(K,K_0)} \circ i_p$$

Now we want to show that the isomorphism i_p takes cycles to cycles and boundaries to boundaries. Let $c_p \in Z_p(L, L_0) = \operatorname{Ker} \partial_p^{(L,L_0)}$. Then $\partial_p^{(L,L_0)} c$ is the identity of $C_{p-1}(L, L_0)$. Isomorphism maps identity to identity, so $i_{p-1}\left(\partial_p^{(L,L_0)} c\right)$ is the identity of $C_{p-1}(K, K_0)$.

Since $i_{p-1} \circ \partial_p^{(L,L_0)} = \partial_p^{(K,K_0)} \circ i_p$, we can conclude that $\partial_p^{(K,K_0)}(i_p c)$ is the identity of $C_{p-1}(K, K_0)$. Therefore, $i_p c \in \operatorname{Ker} \partial_p^{(K,K_0)} = Z_p(K, K_0)$. So i_p takes relative p-cycles of L modulo L_0 to relative p-cycles of K modulo K_0.

Now let $d \in B_p(L, L_0) = \operatorname{im} \partial_{p+1}^{(L,L_0)}$. So $d = \partial_{p+1}^{(L,L_0)} e$ for some $e \in C_{p+1}(L, L_0)$.

$$i_p d = i_p\left(\partial_{p+1}^{(L,L_0)} e\right) = \partial_{p+1}^{(K,K_0)}(i_{p+1} e) \implies i_p d \in \operatorname{im} \partial_{p+1}^{(K,K_0)} = B_p(K, K_0)$$

So i_p takes relative p-boundaries of L modulo L_0 to relative p-boundaries of K modulo K_0. Therefore, i_p is the requied isomorphism of the respective relative homology groups establishing $H_p(L, L_0) \cong H_p(K, K_0)$. ■

§3.2 Exact Homology Sequence

It's time to introduce you to relative homology from a formal viewpoint. The related axioms are called *Eilenberg-Steenrod axioms*.

Definition 3.2.1. Consider a sequence (finite or infinite) of groups and homomorphisms

$$\cdots \longrightarrow A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \longrightarrow \cdots$$

This sequence is said to be **exact at** A_2 if $\operatorname{im} \phi_1 = \operatorname{Ker} \phi_2$. If it is everywhere exact, it is said to be an **exact sequence**. Of course, exactness is not defined at the two ends of the sequence, if they exist.

Several useful facts about exact sequences are listed below. The proofs of these facts are left as an exercise for the reader. Note that, the groups under study are all abelian, and hence we denote by 0 the trivial group consisting of only the identity element 0.

1. $A_1 \xrightarrow{\phi} A_2 \to 0$ is exact if and only if ϕ is an epimorphism (surjective).

2. $0 \to A_1 \xrightarrow{\phi} A_2$ is exact if and only if ϕ is a monomorphism (injective).

3. Suppose the sequence $0 \to A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \to 0$ is exact. Such a sequence is called a **short exact sequence**. Here ϕ is a monomorphism and ψ is an epimorphism.

   Exactness yields $\phi(A_1) = \operatorname{Ker} \psi$. Since $\psi : A_2 \to A_3$ is a surjective homomorphism, by the *First Isomorphism Theorem*,

   $$A_2 / \operatorname{Ker} \psi \cong A_3 \implies A_2/\phi(A_1) \cong A_3$$

   Conversely, if $\psi : A \to B$ is an epimorphism with kernel K, then the sequence

   $$0 \longrightarrow K \xrightarrow{\iota} A \xrightarrow{\psi} B \longrightarrow 0$$

   is exact, where ι is inclusion. Indeed $\iota(K) = K = \operatorname{Ker} \psi$.

4. Suppose the sequence $A_1 \xrightarrow{\alpha} A_2 \xrightarrow{\phi} A_3 \xrightarrow{\beta} A_4$ is exact. Then the following are equivalent:

   (i) α is an epimorphism.

(ii) β is a monomorphism.

(iii) ϕ is the zero homomorphism (ϕ maps all of A_2 to the identity element 0 of A_3).

5. Suppose the sequence $A_1 \xrightarrow{\alpha} A_2 \to A_3 \to A_4 \xrightarrow{\beta} A_5$ is exact. Then the following sequence is also exact :

$$0 \longrightarrow \operatorname{Cok}\alpha \longrightarrow A_3 \longrightarrow \operatorname{Ker}\beta \longrightarrow 0$$

If $f : A \to B$ is a homomorphism, then the *Cokernel* of f is defined as $\operatorname{Cok} f := B/\operatorname{im} f$.

Definition 3.2.2. Consider two sequences of groups and homomorphisms having the same index set :

$$\cdots \to A_1 \to A_2 \to \cdots$$

$$\cdots \to B_1 \to B_2 \to \cdots$$

A **homomorphism** of the first sequence into the second is a family of homomorphisms $\alpha_i : A_i \to B_i$ such that each square of maps

$$\begin{array}{ccc} A_i & \longrightarrow & A_{i+1} \\ \alpha_i \downarrow & & \downarrow \alpha_{i+1} \\ B_i & \longrightarrow & B_{i+1} \end{array}$$

connutes. It is an **isomorphism** if each α_i is an isomorphism.

Example 3.2.1 (Chain Complex and Chain Map)

A chain complex $\mathcal{C}$ is a family $\{C_p, \partial_p\}$ of abelian groups C_p and group homomorphism $\partial_p : C_p \to C_{p-1}$, indexed with integers, such that $\partial_p \circ \partial_{p+1} = 0$ for all p.

$$\cdots \longrightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow \cdots$$

Let $\mathcal{C} = \{C_p, \partial_p\}$ and $\mathcal{C}' = \left\{C'_p, \partial'_p\right\}$ be chain complexes. A **chain map** $\phi : \mathcal{C} \to \mathcal{C}'$ is a family of homomorphism $\phi_p : C_p \to C'_p$ such that

$$\partial'_p \circ \phi_p = \phi_{p-1} \circ \partial_p \ , \quad \forall p \, .$$

In other words, each of the following squares commutes:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{p+1} & \xrightarrow{\partial_{p+1}} & C_p & \xrightarrow{\partial_p} & C_{p-1} & \longrightarrow & \cdots \\ & & \phi_{p+1}\downarrow & & \downarrow\phi_p & & \downarrow\phi_{p-1} & & \\ \cdots & \longrightarrow & C'_{p+1} & \xrightarrow[\partial'_{p+1}]{} & C'_p & \xrightarrow[\partial'_p]{} & C'_{p-1} & \longrightarrow & \cdots \end{array}$$

Definition 3.2.3. Consider the following short exact sequence:

$$0 \longrightarrow A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \longrightarrow 0$$

This sequence is said to be **split** if the group $\phi(A_1)$ is a direct summand in A_2. In other words, A_2 is the direct sum of $\phi(A_1)$ and some other subgroup B of A_2. (The direct sum is termed **internal** as the structure A_2 is known apriori which is written as the direct sum of the summands, as opposed to the case of **external** direct sum where the resulting object is not given apriori.)

We express this fact with the following exact sequence:

$$0 \longrightarrow A_1 \xrightarrow{\phi} \phi(A_1) \oplus B \xrightarrow{\psi} A_3 \longrightarrow 0$$

An equivalent formulation using **external** direct sum is given as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow \theta & & \downarrow = & & \\ 0 & \longrightarrow & A_1 & \xrightarrow[\iota]{} & A_1 \oplus A_3 & \xrightarrow[\pi]{} & A_3 & \longrightarrow & 0 \end{array}$$

In this case, $\oplus$ denotes external direct sum; ι is the inclusion and π is the cannonical projection. The map $\theta : A_2 \to A_1 \oplus A_3$ is defined as follows:

$$\text{For } a \in A_2 \ , \ \theta(a) \equiv ((\theta(a))_1, (\theta(a))_2) = \left(\phi^{-1}(a), \psi(a)\right)$$

4 Lecture 4

§4.1 Exact Homology Sequence Continued

Theorem 4.1.1

Let $0 \to A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \to 0$ be exact. Then the following are equivalent:

1. The sequence splits.
2. There is a map $p : A_2 \to A_1$ such that $p \circ \phi = \mathrm{id}_{A_1}$.
3. There is a map $j : A_3 \to A_2$ such that $\psi \circ j = \mathrm{id}_{A_3}$.

$$0 \longrightarrow A_1 \underset{p}{\overset{\phi}{\rightleftarrows}} A_2 \underset{j}{\overset{\psi}{\rightleftarrows}} A_3 \longrightarrow 0$$

Proof. First, we show that (1)⇒(2) and (1)⇒(3). From the definition of split exact sequence using external direct sum, it follows that it suffices to prove (2) and (3) for the following sequence:

$$0 \longrightarrow A_1 \xrightarrow{\iota} A_1 \oplus A_3 \xrightarrow{\pi} A_3 \longrightarrow 0$$

To prove (2), one takes $p : A_1 \oplus A_3 \to A_1$ to be the projection onto the first factor; and to prove (3), one takes $j : A_3 \to A_1 \oplus A_3$ to be the standard inclusion map.

(2)⇒(1): We show that $A_2 = \phi(A_1) \oplus \operatorname{Ker} p$. It will prove that

$$0 \longrightarrow A_1 \underset{p}{\overset{\phi}{\rightleftarrows}} \underbrace{\phi(A_1) \oplus \operatorname{Ker} p}_{A_2} \xrightarrow{\psi} A_3 \longrightarrow 0$$

is a split exact sequence. First, for $x \in A_2$, we write

$$x = \phi(p(x)) + (x - \phi(p(x)))$$

Since, $p(x) \in A_1$, $\phi(p(x)) \in \phi(A_1)$. On the other hand,

$$p(x - \phi(p(x))) = p(x) - p(\phi(p(x))) = p(x) - \mathrm{id}_{A_1}(p(x)) = 0$$

Therefore, $x - \phi(p(x)) \in \operatorname{Ker} p$. Now we will show that $\phi(A_1) \cap \operatorname{Ker} p$ is trivial.

Let $x \in \phi(A_1) \cap \operatorname{Ker} p$. Then there exists $y \in A_1$ such that $\phi(y) = x$, and $p(x) = 0$.

$$0 = p(x) = p(\phi(y)) = \mathrm{id}_{A_1}(y) = y$$

So $y = 0$. ϕ is a homomorphism, so it maps identity to identity. Therefore, $x = \phi(y) = \phi(0) = 0$, so $\phi(A_1) \cap \operatorname{Ker} p = \{0\}$. Hence, $A_2 = \phi(A_1) \oplus \operatorname{Ker} p$.

(3)⇒(1): We show that $A_2 = \operatorname{Ker} \psi \oplus j(A_3)$. Since $\operatorname{Ker} \psi = \operatorname{im} \phi$, proving $A_2 = \operatorname{Ker} \psi \oplus j(A_3)$ will imply $A_2 = \phi(A_1) \oplus j(A_3)$. It will prove that

$$0 \longrightarrow A_1 \xrightarrow{\phi} \underbrace{\operatorname{Ker} \psi \oplus j(A_3)}_{A_2} \underset{j}{\overset{\psi}{\rightleftarrows}} A_3 \longrightarrow 0$$

is a split exact sequence. For $x \in A_2$, we write

$$x = (x - j(\psi(x))) + j(\psi(x))$$

Since $\psi(x) \in A_3$, $j(\psi(x)) \in j(A_3)$. On the other hand,

$$\psi(x - j(\psi(x))) = \psi(x) - \psi(j(\psi(x))) = \psi(x) - \mathrm{id}_{A_3}(\psi(x)) = 0$$

Therefore, $x - j(\psi(x)) \in \operatorname{Ker}\psi$. Now we will show that $\operatorname{Ker}\psi \cap j(A_3)$ is trivial.

Let $x \in \operatorname{Ker}\psi \cap j(A_3)$. Then there exists $z \in A_3$ such that $j(z) = x$, and $\psi(x) = 0$.

$$0 = \psi(x) = \psi(j(z)) = \mathrm{id}_{A_3}(z) = z$$

So $z = 0$. j is a homomorphism, so it maps identity to identity. Therefore, $x = j(z) = j(0) = 0$. So $\operatorname{Ker}\psi \cap j(A_3) = \{0\}$. Hence, $A_2 = \operatorname{Ker}\psi \oplus j(A_3)$. ■

Corollary 4.1.2

Let $0 \to A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \to 0$ be exact. If A_3 is free abelian, the sequence splits.

Proof. We choose a basis $\{e_\alpha\}_\alpha$ for A_3. We want to show that part (3) of Theorem 4.1.1 holds.

$$0 \longrightarrow A_1 \xrightarrow{\phi} A_2 \underset{j}{\overset{\psi}{\rightleftarrows}} A_3 \longrightarrow 0$$

We define the map $j : A_3 \to A_2$ with the help of its action on the basis elements e_α of A_3. We let $j(e_\alpha)$ be any element of the nonempty set $\psi^{-1}(e_\alpha)$. $\psi^{-1}(e_\alpha)$ is nonempty, because ψ is an epimorphism. Then $\psi(j(e_\alpha)) = e_\alpha$. Therefore, for a given $\sum_\alpha n_\alpha e_\alpha \in A_3$,

$$\psi\left(j\left(\sum_\alpha n_\alpha e_\alpha\right)\right) = \sum_\alpha n_\alpha \psi(j(e_\alpha)) = \sum_\alpha n_\alpha e_\alpha$$

In other words, $\psi \circ j = \mathrm{id}_{A_3}$. Hence, by Theorem 4.1.1, the sequence $0 \to A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \to 0$ splits. ■

§4.2 Simplicial Maps and Simplicial Approximation

Recall that (from Algebraic Topology I), a simplicial map $f : |K| \to |L|$ is a continuous map between the underlying spaces of two simplicial complexes K and L with $g : K^{(0)} \to L^{(0)}$ being a map between the sets of vertices of the complexes such that whenever $v_0, v_1, \ldots, v_n$ span a simplex of K, the points $g(v_0), g(v_1), \ldots, g(v_n)$ are vertices of a simplex of L. For a given $x \in |K|$, the continuous map $f : |K| \to |L|$ satisfies

$$x = \sum_{i=0}^{n} t_i v_i \implies f(x) = \sum_{i=0}^{n} t_i g(v_i)\ .$$

Definition 4.2.1. Let $f : |K| \to |L|$ be a simplicial map. If $[v_0, v_1, \ldots, v_p]$ is an oriented p-simplex of K, then the points $f(v_0), f(v_1), \ldots, f(v_p)$ span a simplex of L. We define a homomorphism $(f_\#)_p : C_p(K) \to C_p(L)$ by defining it on elementary p-chains corresponding to oriented p-simplices of K:

$$(f_\#)_p([v_0, \ldots, v_p]) = \begin{cases} [f(v_0), \ldots, f(v_p)] & \text{if } f(v_0), f(v_1), \ldots, f(v_p) \text{ are distinct} \\ 0 & \text{otherwise} \end{cases}$$

Lemma 4.2.1

The homomosphism $f_\#$ commutes with the boundary operator, *i.e.*

$$\partial_p (f_\#)_p([v_0, \ldots, v_p]) = (f_\#)_{p-1}(\partial_p [v_0, \ldots, v_p])$$

In other words, each of the squares of the following diagram commutes:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{p+1}(K) & \xrightarrow{\partial_{p+1}} & C_p(K) & \xrightarrow{\partial_p} & C_{p-1}(K) & \xrightarrow{\partial_{p-1}} & \cdots \\ & & \downarrow (f_\#)_{p+1} & & \downarrow (f_\#)_p & & \downarrow (f_\#)_{p-1} & & \\ \cdots & \longrightarrow & C_{p+1}(L) & \xrightarrow{\partial_{p+1}} & C_p(L) & \xrightarrow{\partial_p} & C_{p-1}(L) & \xrightarrow{\partial_{p-1}} & \cdots \end{array}$$

Therefore, $f_\#$ induces a homomorphism $(f_*)_p : H_p(K) \to H_p(L)$.

We shall omit the proof. The interested readers are encouraged to go over §12 of the textbook *Elements of Algebraic Topology* by James Munkres.

Lemma 4.2.2

The chain map $f_\#$ preserves the augmentation map ϵ; therefore, it induces a homomorphism f_* of reduced homology groups.

Definition 4.2.2. Let $h : |K| \to |L|$ be a continuous map. We say that h satisfies the **star condition** with respect to K and L if for every vertex v of K there exists a vertex w of L such that

$$h(\operatorname{St} v) \subseteq \operatorname{St} w$$

Definition 4.2.3 (Simplicial Approximation)**.** Let $h : |K| \to |L|$ be a continuous map. If $f : |K| \to |L|$ is a simplicial map such that

$$h(\operatorname{St} v) \subseteq \operatorname{St} f(v) ,$$

we call f a **simplicial approximation** to h.

Remark. $h(\operatorname{St} v) \subseteq \operatorname{St} f(v)$ implies that h satisfies the star condition relative to K and L. In other words, whenever a continuous map $h : |K| \to |L|$ satisfies the star condition, one can find its simplicial approximation; *i.e.* a simplicial map $f : |K| \to |L|$ satisfying $h(\operatorname{St} v) \subseteq \operatorname{St} f(v)$.

Since any simplicial map $f : |K| \to |L|$ induces a homoomrphism $(f_*)_p : H_p(K) \to H_p(L)$, for *any* continuous map $h : |K| \to |L|$ satisfying the star condition relative to K and L, there is a well-defined homomorphism

$$(h_*)_p : H_p(K) \to H_p(L)$$

obtained by setting $h_* = f_*$, where f is the simplicial approximation to h.

Now the problem we are confronted with is that any continuous map $h : |K| \to |L|$ may not satisfy the star condition relative to K and L. There is a useful technique called "subdivision" by means of which one canform a new simplex K' out of the complex K with the same underlying space, *i.e.* $|K| = |K'|$, such that $h : |K| \to |L|$ satisfies the star condition relative to K' and L.

Definition 4.2.4 (Subdivision)**.** Let K be a geometric complex in $\mathbb{E}^J$. A complex K' is said to be a **subdivision** of K if

1. Each simplex of K' is contained in a simplex of K.
2. Each simplex of K equals the union of finitely many simplices of K'.

These two conditions imply that the union of the simplices of K' equals the union of simplices of K — that is K and K' are equal as sets. The finiteness condition (2) guarantees that $|K|$ and $|K'|$ are equal as topological spaces (check!).

Definition 4.2.5. Let K be a complex; suppose that L_p is a subdivision of the p-skeleton $K^{(p)}$ of K. Let σ be a $(p+1)$-simplex of K. The set $\operatorname{Bd}\sigma$ is the polytope of a subcomplex of $K^{(p)}$, and hence of a subcomplex of L_p; we denote the latter subcomplex by L_σ. If w_σ, is an interior point of σ, then the cone $w_\sigma * L_\sigma$ is a complex whose underlying space is σ. We define L_{p+1}, to be the union of L_p and the complexes $w_\sigma * L_\sigma$ as σ ranges over all $(p+1)$-simplices of K. One can show that L_{p+1} is a complex; it is said to be the subdivision of $K^{(p+1)}$ obtained by starring L_p from the points w_σ.

Definition 4.2.6 (Barycenter). Let $\sigma = v_0 \cdots v_p$ be an unoriented p-simlex with the given vertices. The **barycenter** of σ is defined to be the point

$$\hat{\sigma} = \sum_{i=0}^{p} \frac{1}{p+1} v_i \,.$$

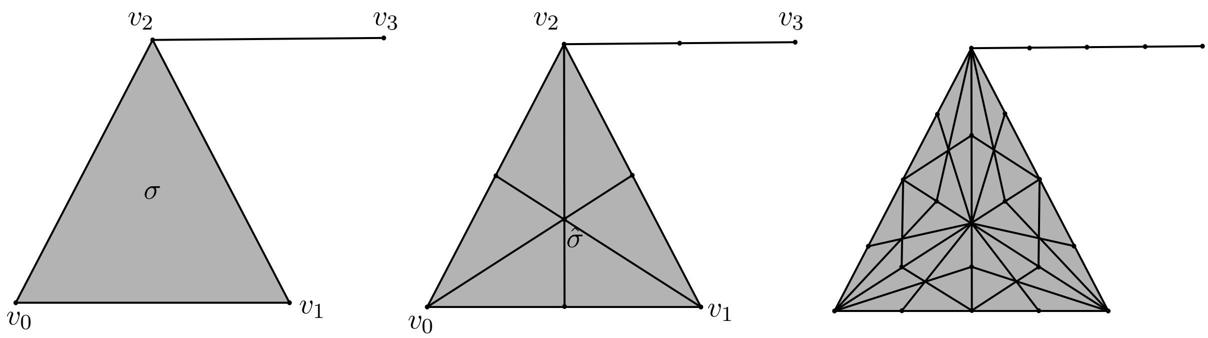
Observe that the barycentric coordinates of $\hat{\sigma}$ with respect to $v_0, \ldots, v_p$ are all equal. And the sum of all these coordinates is $(p+1)\ \frac{1}{p+1} = 1$, as it should be. In other words, the weight of the barycenter $\hat{\sigma}$ on each of the vertices $v_0, \ldots, v_p$ is given by $\frac{1}{p+1}$.

Definition 4.2.7 (Barycentric Subdivision). Let K be a complex. We define a sequence of subdivisions of the skeletons of K as follows: Let $L_0 = K^{(0)}$, the 0-skeleton of K. In general, if L_p is a subdivision of the p-skeleton of K, let L_{p+1} be the subdivision of the $(p+1)$-skeleton obtained by starring L_p from the barycenters of the $(p+1)$-simplices of K.

The union of the complexes L_p (for $p = 0, 1, 2, \ldots$) can be seen to be a subdivision of K using standard results. It is called the **first barycentric subdivision** of K, and denoted by $\operatorname{sd} K$.

Having formed a complex $\operatorname{sd} K$, we can now construct its first barycentric subdivision $\operatorname{sd}(\operatorname{sd} K)$, which we denote by $\operatorname{sd}^2 K$. This complex is called the **second barycentric subdivision** of K. Similarly, one defines $\operatorname{sd}^n K$, in general.

The following image illuatrates a 2-d complex K and its first and second barycentric subdivision.



K $\qquad$ $\operatorname{sd} K$ $\qquad$ $\operatorname{sd}^2 K$

Theorem 4.2.3 (Finite Simplicial Approximation Theorem)

Let K and L be complexes; and let K be finite. Given a contiuous map $h : |K| \to |L|$, there is an $N \in \mathbb{N}$ such that h has a simplicial approximation $f : \operatorname{sd}^N K \to L$.

We are not stating the proof of Theorem 4.2.3 here. To prove it, one needs to show that $h : |K| \to |L|$ satisfies the star condition relative to $\operatorname{sd}^N K$ and L.

Let K_0 be a subcomplex of K, and L_0 be a subcomplex of L. Let $f : K \to L$ be a simplicial map that carries each simplex of K_0 to a simplex of L_0. We often express it as "$f : (K, K_0) \to (L, L_0)$ is a simplicial map". This induces a map $(f_*)_p : H_p(K, K_0) \to H_p(L, L_0)$.

Theorem 4.2.4 (General Simplicial Approximation Theorem)
Let K and L be complexes. Given a contiuous map $h : |K| \to |L|$, there exists a subdivision K' of K such that h has a simplicial approximation $f : K' \to L$.

§4.3 Homology Boundary Homomorphism

First we need to define a homomrphism $(\partial_*)_p : H_p(K, K_0) \to H_{p-1}(K_0)$ that is induced by the boundary operator and is called the **homology boundary homomorphism**. The construction of ∂_* is as follows:

Given a relative p-chain $z \in C_p(K, K_0)$, one can find a p-chain d carried by $K \setminus K_0$ satisfying

$$\{z\} = \underbrace{d + C_p(K_0)}_{\text{coset}}$$

If, in addition, z is a relative p-cycle of K mod K_0, then $\partial_p d$ is carried by K_0. Now consider the inclusions (simplicial maps)

$$i : K_0 \to K \quad \text{and} \quad \pi : (K, \varnothing) \to (K, K_0)$$

Now the induced chain map $(i_\#)_p : C_p(K_0) \to C_p(K)$ is inclusion; and $(\pi_\#)_p : C_p(K) \to C_p(K, K_0)$ is projection map.

Given a relative p-cycle $z \in C_p(K, K_0)$, the chain d of $C_p(K)$ carried by $K \setminus K_0$ such that $(\pi_\#)_p(d) = z$. Since z is a relative p-cycle of K mod K_0, then $\partial_p d$ is carried by K_0, *i.e.* $\partial_p d \in C_{p-1}(K_0)$.

$(i_\#)_{p-1} : C_{p-1}(K_0) \to C_{p-1}(K)$ is the inclusion map, so there exists a $(p-1)$-chain c of K_0 such that $(i_\#)_{p-1}(c) = \partial_p d$.

$$\begin{array}{ccc} C_{p-1}(K_0) & \xrightarrow{(i_\#)_{p-1}} & C_{p-1}(K) \\ \Big\downarrow{\scriptstyle \partial^{K_0}_{p-1}} & & \Big\downarrow{\scriptstyle \partial^{K}_{p-1}} \\ C_{p-2}(K_0) & \xrightarrow{(i_\#)_{p-2}} & C_{p-2}(K) \end{array}$$

The above diagram is easily seen to commute, *i.e.*

$$(i_\#)_{p-2} \circ \partial^{K_0}_{p-1} = \partial^{K}_{p-1} \circ (i_\#)_{p-1}$$

c is a $(p-1)$-chain on K_0, so

$$(i_\#)_{p-2}\left(\partial^{K_0}_{p-1} c\right) = \partial^{K}_{p-1}\left((i_\#)_{p-1} c\right) = \partial^{K}_{p-1}\left(\partial^{K}_{p} d\right) = 0$$

$(i_\#)_{p-2}$ is inclusion, so it's injective. Hence, $\partial^{K_0}_{p-1} c = 0$, *i.e.* the $(p-1)$-chain c is a $(p-1)$-cycle of K_0.

$$\begin{array}{ccccc} & & C_p(K) & \xrightarrow{(\pi_\#)_p} & C_p(K, K_0) \ni z \\ & & \Big\downarrow{\scriptstyle \partial^{K}_{p}} & & \\ c \in C_{p-1}(K_0) & \xrightarrow{(i_\#)_{p-1}} & C_{p-1}(K) & & \end{array}$$

Let us now denote by $\widetilde{c}$ a $(p-1)$-cycle that is not a p-boundary, so that $\{\widetilde{c}\} \in H_{p-1}(K_0)$. Here $\{\widetilde{c}\}$ is an equivalence class: $\widetilde{c} \sim \widetilde{c}'$ iff there exists p-chain e on K_0 such that $\widetilde{c} = \widetilde{c}' + \partial_p e$.

On the other extreme, let us denote by $\widetilde{z}$ a relative p-cycle of K mod K_0 that is not a relative p-boundary of K mod K_0, so that $\{\widetilde{z}\} \in H_p(K, K_0)$. $\{\widetilde{z}\}$ is an equivalence class: $\widetilde{z} \sim \widetilde{z}'$ iff $\widetilde{z} = \widetilde{z}' + \kappa$ with κ being a relative p-boundary of K mod K_0.

We define $(\partial_*)_p : H_p(K, K_0) \to H_{p-1}(K_0)$ by

$$(\partial_*)_p(\{\tilde{z}\}) = \{\tilde{c}\}$$

Now we can state our basic theorem relating the homology of K, K_0 and (K, K_0).

Definition 4.3.1 (Long Exact Sequence)**.** A **long exact sequence** is an exact sequence whose index set is the set of integers. That is, it is a sequence that is infinite in both directions. It may, however, begin or end with an infinite string of trivial groups.

Theorem 4.3.1 (The exact homology sequence of a pair)

Let K be a complex; let K_0 be a subcomplex. Then there is a long exact sequence

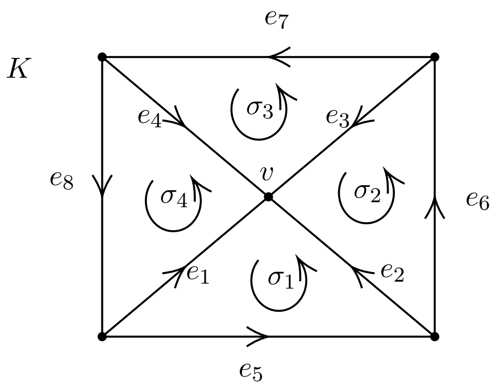
$$\cdots \longrightarrow H_p(K_0) \xrightarrow{(i_*)_p} H_p(K) \xrightarrow{(\pi_*)_p} H_p(K, K_0) \xrightarrow{(\partial_*)_p} H_{p-1}(K_0) \longrightarrow \cdots$$

where $i : K_0 \to K$ and $\pi : (K, \varnothing) \to (K, K_0)$ are inclusions, and ∂_* is induced by the boundary operator. There is a similar exact sequence in reduced homology:

$$\cdots \longrightarrow \widetilde{H_p}(K_0) \longrightarrow \widetilde{H_p}(K) \longrightarrow H_p(K, K_0) \longrightarrow \widetilde{H_{p-1}}(K_0) \longrightarrow \cdots$$

Let us give an example before proving the theorem.

Example 4.3.1. Let K be the following complex whose underlying space is a square; and let K_0 be the subcomplex of K whose undelying space is the boundary of the square. The 2-simplices σ_i are oriented in counterclockwise direction.



We've seen in Example 3.1.3 that $H_2(K, K_0) \cong \mathbb{Z}$. The group $H_2(K, K_0)$ is generated by the 2-chain $\gamma = \sum_{i=1}^{4} \sigma_i$, where the elementary 2-chains corresponding to the oriented 2-simplices are denoted by the same symbol σ_i.

Check that $H_1(K_0) \cong \mathbb{Z}$ and it is generated by $e_5 + e_6 + e_7 + e_8$. In the course of computation of $H_1(K_0)$, one finds that $e_5 + e_6 + e_7 + e_8 = \partial_2 \gamma$. Therefore, in this particular homomorphism, the boundary homomorphism $(\partial_*)_2 : H_2(K, K_0) \to H_1(K_0)$ happens to be $\partial_* : \mathbb{Z} \to \mathbb{Z}$, which is an isomorphism.

$$\begin{array}{l} \cdots \longrightarrow H_2(K) \longrightarrow H_2(K, K_0) \\ \quad\quad\quad\quad (\partial_*)_2 \swarrow \\ H_1(K_0) \longrightarrow H_1(K) \longrightarrow \cdots \end{array}$$

We can prove the fact that $(\partial_*)_2 : H_2(K, K_0) \to H_1(K_0)$ is an isomorphism by considering the exact homology sequence of the pair (K, K_0). Let us focus on the following portion of the exact homology sequence in question:

$$H_2(K) \longrightarrow H_2(K, K_0) \xrightarrow{(\partial_*)_2} H_1(K_0) \longrightarrow H_1(K)$$

We know that both $H_2(K)$ and $H_1(K)$ vanish. Therefore, we have the following exact sequence

$$0 \longrightarrow H_2(K, K_0) \xrightarrow{(\partial_*)_2} H_1(K_0) \longrightarrow 0$$

Now $\operatorname{Ker}(\partial_*)_2 = 0$, so $(\partial_*)_2$ is injective. Also, $\operatorname{im}(\partial_*)_2 = H_1(K_0)$, so $(\partial_*)_2$ is surjective. Therefore, $(\partial_*)_2$ is a bijective homomorphism, *i.e.* an isomorphism.

§4.4 The Zig-Zag Lemma

Now we shall prove Theorem 4.3.1. We shall reformulate this result as a theorem about chain complexes and prove it in that form.

Definition 4.4.1. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be chain complexes. Let 0 denote the trivial chain complex whose groups vanish in every dimension. Let $\phi : \mathcal{C} \to \mathcal{D}$ and $\psi : \mathcal{D} \to \mathcal{E}$ be chain maps. We say the sequence

$$0 \longrightarrow \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \longrightarrow 0$$

is exact, or that it is a **short exact sequence of chain complexes**, if in each dimension p, the sequence

$$0 \longrightarrow C_p \xrightarrow{\phi_p} D_p \xrightarrow{\psi_p} E_p \longrightarrow 0$$

is an exact sequence of groups.

For example, if K is a complex and K_0 is a subcomplex of K and $\mathcal{C}$ is the chain complex $\mathcal{C} = \{C_p(K), \partial_p\}$, the sequence

$$0 \longrightarrow \mathcal{C}(K_0) \longrightarrow \mathcal{C}(K) \longrightarrow \mathcal{C}(K, K_0) \longrightarrow 0$$

is exact. Because $C_p(K, K_0) = C_p(K)/C_p(K_0)$ by definition.

Lemma 4.4.1 (The Zig-Zag Lemma)
Suppose one is given chain complexes $\mathcal{C} = \{C_p, \partial_p^C\}$, $\mathcal{D} = \{D_p, \partial_p^D\}$ and $\mathcal{E} = \{E_p, \partial_p^E\}$; and chain maps ϕ and ψ such that the sequence

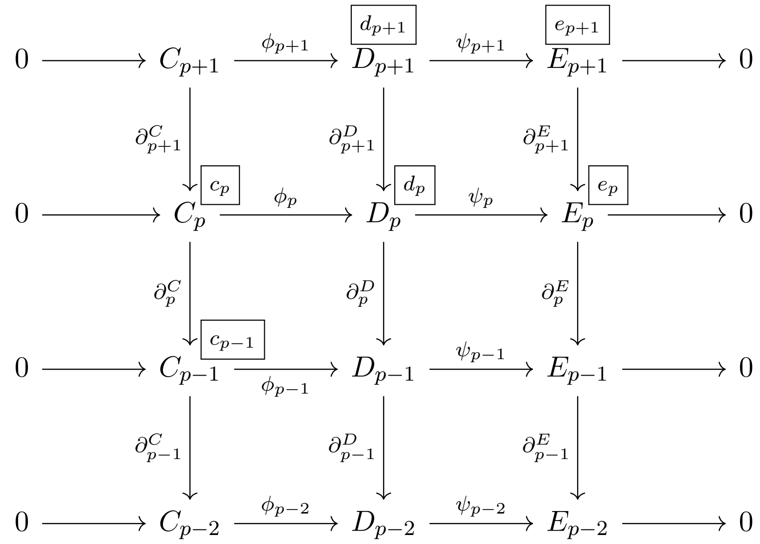
$$0 \longrightarrow \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \longrightarrow 0$$

is exact. Then there is a long exact homology sequence

$$\cdots \longrightarrow H_p(\mathcal{C}) \xrightarrow{(\phi_p)_*} H_p(\mathcal{D}) \xrightarrow{(\psi_p)_*} H_p(\mathcal{E}) \xrightarrow{(\partial_*)_p} H_{p-1}(\mathcal{C}) \xrightarrow{(\phi_{p-1})_*} H_{p-1}(\mathcal{D}) \longrightarrow \cdots$$

where $(\partial_*)_p$ is induced by the boundary operator in $\mathcal{D}$.

Proof. The proof is of a type now commonly known as "diagram-chasing". We shall use the following commutative diagram:



Step 1. First we define $(\partial_*)_p$. Given a cycle $e_p \in E_p$, an element of $\operatorname{Ker} \partial^E_p$, choose $d_p \in D_p$ such that $\psi_p(d_p) = e_p$. This can always be done, because ψ is surjective. Now, the element $\partial^D_p d_p$ of D_{p-1} lies in $\operatorname{Ker} \psi_{p-1}$, because

$$\psi_{p-1}\left(\partial^D_p d_p\right) = \partial^E_p\left(\psi_p\left(d_p\right)\right) = \partial^E_p\left(e_p\right) = 0\,.$$

Now since $\operatorname{Ker} \psi_{p-1} = \operatorname{im} \phi_{p-1}$, there exists $c_{p-1} \in C_{p-1}$ such that $\phi_{p-1}(c_{p-1}) = \partial^D_p d_p$. This element c_{p-1} is unique, since ϕ_{p-1} is injective.

$$\phi_{p-2}\left(\partial^C_{p-1} c_{p-1}\right) = \partial^D_{p-1}\left(\phi_{p-1}\left(c_{p-1}\right)\right) = \partial^D_{p-1}\left(\partial^D_p d_p\right) = 0$$

Since ϕ_{p-2} is injective, $\partial^C_{p-1} c_{p-1} = 0$. Then we define

$$(\partial_*)_p\left(\{e_p\}\right) = \{c_{p-1}\}$$

where $\{\ \}$ denotes the homology class.

Step 2. Now we show that $(\partial_*)_p$ is a well-defined homomorphism. Let $e_p, e'_p \in \operatorname{Ker} \partial^E_p$. Choose $d_p, d'_p \in D_p$ such that $\psi_p(d_p) = e_p$ and $\psi_p\left(d'_p\right) = e'_p$. Then choose $c_{p-1}, c'_{p-1} \in C_{p-1}$ such that $\phi_{p-1}(c_{p-1}) = \partial^D_p d_p$ and $\phi_{p-1}\left(c'_{p-1}\right) = \partial^D_p d'_p$. In this way, we have

$$(\partial_*)_p\left(\{e_p\}\right) = \{c_{p-1}\}\ \ , \ \text{ and } (\partial_*)_p\left(\left\{e'_p\right\}\right) = \left\{c'_{p-1}\right\}$$

To show that $(\partial_*)_p$ is well defined, we suppose e_p and e'_p are homologous, *i.e.* they belong to the same homology class. We need to show that c_{p-1} and c'_{p-1} are also homologous.

Suppose $e_p - e'_p = \partial^E_{p+1} e_{p+1}$. Choose $d_{p+1} \in D_{p+1}$ such that $\psi_{p+1}(d_{p+1}) = e_{p+1}$. This is doable since ψ_{p+1} is surjective. Then

$$\begin{aligned}\psi_p\left(d_p - d'_p - \partial^D_{p+1} d_{p+1}\right) &= e_p - e'_p - \psi_p\left(\partial^D_{p+1} d_{p+1}\right) = e_p - e'_p - \partial^E_{p+1}\left(\psi_{p+1}\left(d_{p+1}\right)\right)\\ &= e_p - e'_p - \partial^E_{p+1} e_{p+1} = 0\end{aligned}$$

So $d_p - d'_p - \partial^D_{p+1} d_{p+1} \in \operatorname{Ker} \psi_p = \operatorname{im} \phi_p$. Hence, $d_p - d'_p - \partial^D_{p+1} d_{p+1} = \phi_p(c_p)$ for some $c_p \in C_p$.

$$\begin{aligned}\phi_{p-1}\left(\partial^C_p c_p\right) &= \partial^D_p\left(\phi_p\left(c_p\right)\right) = \partial^D_p\left(d_p - d'_p - \partial^D_{p+1} d_{p+1}\right)\\ &= \phi_{p-1}\left(c_{p-1}\right) - \phi_{p-1}\left(c'_{p-1}\right) = \phi_{p-1}\left(c_{p-1} - c'_{p-1}\right)\end{aligned}$$

Since ϕ_{p-1} is injective, we have $c_{p-1} - c'_{p-1} = \partial^C_p c_p$. So c_{p-1} and c'_{p-1} are homologous. Therefore, $(\partial_*)_p$ is a well-defined map.

Now we want to show that $(\partial_*)_p$ is a homomorphism. Note that

$$\psi_p\left(d_p + d'_p\right) = e_p + e'_p \text{ and } \phi_{p-1}\left(c_{p-1} + c'_{p-1}\right) = \partial^D_p d_p + \partial^D_p d'_p = \partial^D_p\left(d_p + d'_p\right)$$

Using the definition of $(\partial_*)_p$, one finds that

$$(\partial_*)_p\left(\left\{e_p + e'_p\right\}\right) = \left\{c_{p-1} + c'_{p-1}\right\} = \left\{c_{p-1}\right\} + \left\{c'_{p-1}\right\} = (\partial_*)_p\left(\left\{e_p\right\}\right) + (\partial_*)_p\left(\left\{e'_p\right\}\right)$$

So $(\partial_*)_p$ is a homomorphism.

Step 3. Now we are gonna prove exactness at $H_p(\mathcal{D})$. In other words, we prove $\operatorname{im}(\phi_p)_* = \operatorname{Ker}(\psi_p)_*$. Let $\gamma \in H_p(\mathcal{D})$. Since the sequence

$$0 \longrightarrow \mathcal{C} \xrightarrow{\ \phi\ } \mathcal{D} \xrightarrow{\ \psi\ } \mathcal{E} \longrightarrow 0$$

is exact, we have $\operatorname{Ker}\psi = \operatorname{im}\phi$, *i.e.* $\psi \circ \phi = 0$. Therefore, $\psi_p \circ \phi_p = 0$ for every p. By the **functorial property of induced homomorphism of homology groups**, (see §18 of *Elements of Algebraic Topology* by James R. Munkres for more details)

$$(\psi_p)_* \circ (\phi_p)_* = (\psi_p \circ \phi_p)_* = 0$$

Thus, if $\gamma \in \operatorname{im}(\phi_p)_*$, we have $(\psi_p)_*(\gamma) = 0$. In other words, $\operatorname{im}(\phi_p)_* \subseteq \operatorname{Ker}(\psi_p)_*$.

Now let $\gamma = \{d_p\}$, and suppose $(\psi_p)_*(\gamma) = 0$.

$$0 = (\psi_p)_*(\{d_p\}) = \{\psi_p(d_p)\}$$

So $\{\psi_p(d_p)\}$ is the 0 homology class. In other words, $\psi_p(d_p)$ is homologous to 0. So $\psi_p(d_p) = \partial^E_{p+1} e_{p+1}$ for some $e_{p+1} \in E_{p+1}$. Since ψ_{p+1} is surjective, there exists some $d_{p+1} \in D_{p+1}$ such that $\psi_{p+1}(d_{p+1}) = e_{p+1}$. Then we have

$$\begin{aligned}\psi_p\left(d_p - \partial^D_{p+1} d_{p+1}\right) &= \psi_p(d_p) - \psi_p\left(\partial^D_{p+1} d_{p+1}\right) = \partial^E_{p+1} e_{p+1} - \partial^E_{p+1}\left(\psi_{p+1} d_{p+1}\right)\\ &= \partial^E_{p+1} e_{p+1} - \partial^E_{p+1} e_{p+1} = 0\end{aligned}$$

Therefore, $d_p - \partial^D_{p+1} d_{p+1} \in \operatorname{Ker}\psi_p = \operatorname{im}\phi_p$. In other words, $d_p - \partial^D_{p+1} d_{p+1} = \phi_p(c_p)$ for some $c_p \in C_p$. Now,

$$\phi_{p-1}\left(\partial^C_p c_p\right) = \partial^D_p\left(\phi_p(c_p)\right) = \partial^D_p\left(d_p - \partial^D_{p+1} d_{p+1}\right) = \partial^D_p d_p = 0$$

Since ϕ_{p+1} is injective, we have $\partial^C_p c_p = 0$, which means c_p is a cycle. Furthermore, using the fact that $d_p - \partial^D_{p+1} d_{p+1}$ is homologous to d_p, we get

$$(\phi_p)_*\{c_p\} = \{\phi_p(c_p)\} = \left\{d_p - \partial^D_{p+1} d_{p+1}\right\} = \{d_p\} = \gamma$$

So $\gamma \in \operatorname{im}(\phi_p)_*$. Hence, $\operatorname{Ker}(\psi_p)_* \subseteq \operatorname{im}(\phi_p)_*$. Therefore, $\operatorname{im}(\phi_p)_* = \operatorname{Ker}(\psi_p)_*$, which proves exactness at $H_p(\mathcal{D})$.

Step 4. Now we shall prove exactness at $H_p(\mathcal{E})$. Let $\alpha = \{e_p\} \in H_p(\mathcal{E}) = \operatorname{Ker}\partial^E_p / \operatorname{im}\partial_{p+1}E$. In particular, $e_p \in E_p$. And ψ_p is surjective. So we can choose $d_p \in D_p$ such that $\psi_p(d_p) = e_p$. Then choose $c_{p-1} \in C_{p-1}$ such that $\phi_{p-1}(c_{p-1}) = \partial^D_p d_p$. By the definition of $(\partial_*)_p$,

$$(\partial_*)_p\,\alpha = (\partial_*)_p\{e_p\} = \{c_{p-1}\}$$

Let $\alpha \in \operatorname{im}(\psi_p)_*$, we need to show that $\alpha \in \operatorname{Ker}(\partial_*)_p$. Since $\alpha \in \operatorname{im}(\psi_p)_*$, $\alpha = (\psi_p)_*\gamma$ for some $\gamma = \{d_p\} \in H_p(\mathcal{D})$, where d_p is a cycle. In other words,

$$\{e_p\} = \alpha = (\psi_p)_*\{d_p\} = \{\psi_p(d_p)\}$$

Now, $\phi_{p-1}(c_{p-1}) = \partial^D_p d_p = 0$, and ϕ_{p-1} is injective, so $c_{p-1} = 0$. Therefore, $(\partial_*)_p\,\alpha = \{c_{p-1}\} = 0$. Hence, $\operatorname{im}(\psi_p)_* \subseteq \operatorname{Ker}(\partial_*)_p$.

Now let $\alpha \in \operatorname{Ker}(\partial_*)_p$, and we need to show that $\alpha \in \operatorname{im}(\psi_p)_*$.

$$0 = (\partial_*)_p\,\alpha = \{c_{p-1}\}$$

So c_{p-1} is homologous to 0, *i.e.* $c_{p-1} = \partial_p^C c_p$ for some $c_p \in C_p$. Now we claim that $d_p - \phi_p(c_p)$ is a cycle.

$$\begin{aligned}\partial_p^D (d_p - \phi_p(c_p)) &= \partial_p^D d_p - \partial_p^D (\phi_p(c_p)) = \partial_p^D d_p - \phi_{p-1}\left(\partial_p^C c_p\right) \\ &= \phi_{p-1}(c_{p-1}) - \phi_{p-1}(c_{p-1}) = 0\end{aligned}$$

So $d_p - \phi_p(c_p)$ is a cycle. Now,

$$(\psi_p)_* \{d_p - \phi_p(c_p)\} = \{\psi_p(d_p) - \psi_p(\phi_p(c_p))\} = \{\psi_p(d_p)\} = \{e_p\} = \alpha$$

Therefore, $\alpha \in \operatorname{im}(\psi_p)_*$, so $\operatorname{Ker}(\partial_*)_p \subseteq (\psi_p)_*$. Hence, $\operatorname{Ker}(\partial_*)_p = (\psi_p)_*$.

Step 5. Finally we prove exactness at $H_{p-1}(\mathcal{C})$. Let $\beta = \{c_{p-1}\} \in H_{p-1}(\mathcal{C})$. Suppose $\beta \in \operatorname{im}(\partial_*)_p$. Then $\{c_{p-1}\} = (\partial_*)_p \{e_p\}$ for some $\{e_p\} \in H_p(\mathcal{E})$.

By the definition of $(\partial_*)_p$, $\psi_p(d_p) = e_p$ for some $d_p \in D_p$, and $\phi_{p-1}(c_{p-1}) = \partial_p^D d_p$.

$$(\phi_{p-1})_* \beta = (\phi_{p-1})_* \{c_{p-1}\} = \left\{\partial_p^D d_p\right\} = 0$$

Therefore, $\beta \in \operatorname{Ker}(\phi_{p-1})_*$. So $\operatorname{im}(\partial_*)_p \subseteq \operatorname{Ker}(\phi_{p-1})_*$.

Now let $\beta \in \operatorname{Ker}(\phi_{p-1})_*$. Then we have

$$0 = (\phi_{p-1})_* \beta = (\phi_{p-1})_* \{c_{p-1}\} = \{\phi_{p-1}(c_{p-1})\}$$

So $\phi_{p-1}(c_{p-1}) = \partial_p^D d_p$ for some $d_p \in D_p$. We define $e_p = \psi_p(d_p)$. Then

$$\partial_p^E e_p = \partial_p^E (\psi_p(d_p)) = \psi_{p-1}\left(\partial_p^D d_p\right) = \psi_{p-1}(\phi_{p-1}(c_{p-1})) = 0$$

because $\psi_{p-1} \circ \phi_{p-1} = 0$. Therefore, e_p is a cycle. Using the definition of $(\partial_*)_p$, we get

$$(\partial_*)_p \{e_p\} = \{c_{p-1}\} = \beta$$

So $\beta \in \operatorname{im}(\partial_*)_p$. In other words, $\operatorname{Ker}(\phi_{p-1})_* \subseteq \operatorname{im}(\partial_*)_p$. Therefore, $\operatorname{Ker}(\phi_{p-1})_* = \operatorname{im}(\partial_*)_p$, and thus the sequence is exact at $H_{p-1}(\mathcal{C})$. ■

5 Lecture 5

Suppose $\mathcal{C} = \{C_p, \partial_p\}$ and $\mathcal{C}' = \left\{C'_p, \partial'_p\right\}$ are chain complexes and $\phi : \mathcal{C} \to \mathcal{C}'$ is a chain map. Then this chain map induces a homomorphism at the level of homology groups:

$$(\phi_*)_p : H_p(\mathcal{C}) \to H_p(\mathcal{C}')$$

Also, a chain complex $\mathcal{C}$ is a family $\{C_p, \partial_p\}$ of abelian groups C_p and homomorphisms $\partial_p : C_p \to C_{p+1}$, indexed with integers, such that $\partial_p \circ \partial_{p+1} = 0$ for each p.

If $C_p = 0$ for every $p < 0$, then $\mathcal{C}$ is said to be a **non-negative chain complex**. If C_p is free abelian for each p, then $\mathcal{C}$ is called a **free chain complex**. The group $H_p(\mathcal{C}) = \operatorname{Ker}\partial_p / \operatorname{im}\partial_{p+1}$ is called the ***p*-th homology group** of $\mathcal{C}$.

If $\mathcal{C}$ is a non-negative chain complex, an augmentation for $\mathcal{C}$ is an epimorphism $\epsilon : C_0 \to \mathbb{Z}$ such that $\epsilon \circ \partial_1 = 0$.

$$\cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbb{Z}$$

Then $\operatorname{im}\partial_1 \subseteq \operatorname{Ker}\epsilon$. The **augmented chain complex** $\{\mathcal{C}, \epsilon\}$ is obtained from $\mathcal{C}$ by adjoining the group $\mathbb{Z}$ in dimension -1, and using ϵ as the boundary operator in dimension 0. The homology groups of the augmented chain complex are called **reduced homology groups** of the original chain complex. They are denoted by either $H_i(\{\mathcal{C}, \epsilon\})$ or $\widetilde{H}_i(\mathcal{C})$. Then one has

$$H_p(\mathcal{C}) = \begin{cases} \widetilde{H}_p(\mathcal{C}) & \text{for every } p \neq 0 \\ \widetilde{H}_0(\mathcal{C}) \oplus \mathbb{Z} & \text{for } p = 0 \end{cases}$$

If $\phi : \mathcal{C} \to \mathcal{C}'$ and $\psi : \mathcal{C}' \to \mathcal{C}''$ are chain maps, then $\psi \circ \phi : \mathcal{C} \to \mathcal{C}''$ is also a chain map. T he induced homomorphism of $\psi \circ \phi$ reads $(\psi \circ \phi)_* = \psi_* \circ \phi_*$. In other words,

$$(\psi_p \circ \phi_p)_* = (\psi_*)_p \circ (\phi_*)_p$$

If $\{\mathcal{C}, \epsilon\}$ and $\{\mathcal{C}', \epsilon'\}$ are augmented chain complexes, the chain map $\phi : \mathcal{C} \to \mathcal{C}'$ is said to be **augmentation preserving** if $\epsilon' \circ \phi_0 = \epsilon$. If we extened ϕ to the (-1)-dimensional groups by letting ϕ_{-1} equal the identity map of $\mathbb{Z}$, then ϕ is called a **chain map of augmented chain complexes**.

$$\begin{array}{ccccccc} \cdots \longrightarrow & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\epsilon} & \mathbb{Z} \\ & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_{-1}=\mathrm{id}} \\ \cdots \longrightarrow & C'_1 & \xrightarrow{\partial'_1} & C'_0 & \xrightarrow{\epsilon'} & \mathbb{Z} \end{array}$$

Here, commutativity of the rightmost square implies that $\epsilon' \circ \phi_0 = \epsilon \circ \mathrm{id} = \epsilon$.

It follows that an augmentation preserving chain map ϕ induces a homomorphism

$$(\phi_*)_p : \widetilde{H}_p(\mathcal{C}) \to \widetilde{H}_p(\mathcal{C}')$$

of reduced homology groups.

§5.1 Application of Exact Homology Sequences

Theorem 5.1.1

Suppose one is given the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{C} & \xrightarrow{\phi} & \mathcal{D} & \xrightarrow{\psi} & \mathcal{E} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & \mathcal{C}' & \xrightarrow{\phi'} & \mathcal{D}' & \xrightarrow{\psi'} & \mathcal{E}' & \longrightarrow & 0 \end{array}$$

where the horizontal sequences are exact sequences of chain complexes; and α, β, γ are chain maps. Then the following diagram commutes as well:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_p(\mathcal{C}) & \xrightarrow{(\phi_p)_*} & H_p(\mathcal{D}) & \xrightarrow{(\psi_p)_*} & H_p(\mathcal{E}) & \xrightarrow{(\partial_*)_p} & H_{p-1}(\mathcal{C}) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle(\alpha_p)_*} & & \downarrow{\scriptstyle(\beta_p)_*} & & \downarrow{\scriptstyle(\gamma_p)_*} & & \downarrow{\scriptstyle(\alpha_{p-1})_*} & & \\ \cdots & \longrightarrow & H_p(\mathcal{C}') & \xrightarrow[(\phi'_p)_*]{} & H_p(\mathcal{D}') & \xrightarrow[(\psi'_p)_*]{} & H_p(\mathcal{E}') & \xrightarrow[(\partial'_*)_p]{} & H_{p-1}(\mathcal{C}') & \longrightarrow & \cdots \end{array}$$

Proof. We know that the chain complexes $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are pairs $\left(C_p, \partial_p^C\right)$, $\left(D_p, \partial_p^D\right)$, $\left(E_p, \partial_p^E\right)$ with C_p, D_p, E_p being abelian groups. Since $\phi, \psi, \alpha, \beta, \gamma$ are all chain maps, the commutative diagram above yields the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_p & \xrightarrow{\phi_p} & D_p & \xrightarrow{\psi_p} & E_p & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha_p} & & \downarrow{\scriptstyle\beta_p} & & \downarrow{\scriptstyle\gamma_p} & & \\ 0 & \longrightarrow & C'_p & \xrightarrow{\phi'_p} & \mathcal{D}' & \xrightarrow{\psi'_p} & E'_p & \longrightarrow & 0 \end{array}$$

In other words, $\beta_p \circ \phi_p = \phi'_p \circ \alpha_p$, and $\gamma_p \circ \psi_p = \psi'_p \circ \beta_p$. From the compositional properties of induced homomorphism, it follows that

$$(\beta_p)_* \circ (\phi_p)_* = (\beta_p \circ \phi_p)_* = \left(\phi'_p \circ \alpha_p\right)_* = \left(\phi'_p\right)_* \circ (\alpha_p)_*$$

Similarly, we have

$$(\gamma_p)_* \circ (\psi_p)_* = \left(\psi'_p\right)_* \circ (\beta_p)_*$$

These two altogether gives us the following commutative diagram:

$$\begin{array}{ccccc} H_p(\mathcal{C}) & \xrightarrow{(\phi_p)_*} & H_p(\mathcal{D}) & \xrightarrow{(\psi_p)_*} & H_p(\mathcal{E}) \\ \downarrow{\scriptstyle(\alpha_p)_*} & & \downarrow{\scriptstyle(\beta_p)_*} & & \downarrow{\scriptstyle(\gamma_p)_*} \\ H_p(\mathcal{C}') & \xrightarrow[(\phi'_p)_*]{} & H_p(\mathcal{D}') & \xrightarrow[(\psi_p)_*]{} & H_p(\mathcal{E}') \end{array}$$

So, the first two squares of our desired commutative diagram are shown to commute.

Now let us examin the definition of $(\partial_*)_p$ and $(\partial'_*)_p$ in the light of what we studied in the previous chapter.

$$\begin{array}{ccccccccc} & & & & \boxed{d_p} & & \boxed{e_p} & & \\ 0 & \longrightarrow & C_p & \xrightarrow{\phi_p} & D_p & \xrightarrow{\psi_p} & E_p & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\partial_p^C} & & \downarrow{\scriptstyle\partial_p^D} & & \downarrow{\scriptstyle\partial_p^E} & & \\ 0 & \longrightarrow & C_{p-1} & \xrightarrow{\phi_{p-1}} & D_{p-1} & \xrightarrow{\psi_{p-1}} & E_{p-1} & \longrightarrow & 0 \\ & & \boxed{c_{p-1}} & & \boxed{\partial_p^D d_p} & & & & \end{array}$$

Given $\{e_p\} \in H_p(\mathcal{E})$, we choose $d_p \in D_p$ such that $\psi_p(d_p) = e_p$ (because ψ_p is surjective). Now, $\partial_p^D d_p$ lies in $\operatorname{Ker}\psi_{p-1} = \operatorname{im}\phi_{p-1}$, because

$$\psi_{p-1}\left(\partial_p^D d_p\right) = \partial_p^E(\psi_p d_p) = \partial_p^E(e_p) = 0$$

So, there exists $c_{p-1} \in C_{p-1}$ such that $\phi_{p-1}(c_{p-1}) = \partial_p^D d_p$. c_{p-1} is a cycle, as proved in previous chapter. By the definition of $(\partial_*)_p$, one has

$$(\partial_*)_p\{e_p\} = \{c_{p-1}\}$$

Now let $e'_p = \gamma_p(e_p)$. This means $(\gamma_p)_*\{e_p\} = \{\gamma_p(e_p)\} = \{e'_p\}$. We need to show that

$$\left((\partial'_*)_p \circ (\gamma_p)_*\right)\{e_p\} = \left((\alpha_{p-1})_* \circ (\partial_*)_p\right)\{e_p\},$$

which is equivalent to showing

$$(\partial'_*)_p\{e'_p\} = (\alpha_{p-1})_*\{c_{p-1}\}.$$

If we go back to the first commutative diagram of this proof, commutativity of the right hand square gives us

$$\psi'_p(\beta_p(d_p)) = \gamma_p(\psi_p(d_p)) = \gamma_p(e_p) = e'_p$$

So $\beta_p(d_p)$ is a suitable pullback of e'_p in D'_p. Now, commutativity of the left hand square in the same diagram (for $p-1$) gives us

$$\phi'_{p-1}(\alpha_{p-1}(c_{p-1})) = \beta_{p-1}(\phi_{p-1}(c_{p-1})) = \beta_{p-1}\left(\partial_p^D d_p\right) = \partial_p^{D'}(\beta_p(d_p))$$

The last equality follows from the following commutative diagram:

$$\begin{array}{ccc} D_p & \xrightarrow{\beta_p} & D'_p \\ \partial_p^D \big\downarrow & & \big\downarrow \partial_p^{D'} \\ D_{p-1} & \xrightarrow{\beta_{p-1}} & D'_{p-1} \end{array}$$

Therefore, we get that $\alpha_{p-1}(c_{p-1})$ is a suitable pullback of $\partial_p^{D'}(\beta_p(d_p))$ in C'_{p-1}.

$$\begin{array}{ccccccccc} & & & & \boxed{\beta_p(d_p)} & & \boxed{e'_p} & & \\ 0 & \longrightarrow & C'_p & \xrightarrow{\phi'_p} & D'_p & \xrightarrow{\psi'_p} & E'_p & \longrightarrow & 0 \\ & & \partial_p^{C'} \big\downarrow & & \partial_p^{D'} \big\downarrow & & \big\downarrow \partial_p^{E'} & & \\ 0 & \longrightarrow & C'_{p-1} & \xrightarrow{\phi'_{p-1}} & D'_{p-1} & \xrightarrow{\psi'_{p-1}} & E'_{p-1} & \longrightarrow & 0 \\ & & \boxed{\alpha_{p-1}(c_{p-1})} & & \boxed{\partial_p^{D'}(\beta_p(d_p))} & & & & \end{array}$$

And $\alpha_{p-1}(c_{p-1})$ is a cycle since chain maps take cycles to cycles. Therefore, we get

$$(\partial'_*)_p\{e'_p\} = \{\alpha_{p-1}(c_{p-1})\} = (\alpha_p)_*\{c_{p-1}\}$$

Therefore, we are done! ■

Lemma 5.1.2 (Steenrod Five Lemma)

Suppose one is given the following commutative diagram of abelian groups and homomorphisms:

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{g_1} & A_2 & \xrightarrow{g_2} & A_3 & \xrightarrow{g_3} & A_4 & \xrightarrow{g_4} & A_5 \\ \big\downarrow f_1 & & \big\downarrow f_2 & & \big\downarrow f_3 & & \big\downarrow f_4 & & \big\downarrow f_5 \\ B_1 & \xrightarrow{h_1} & B_2 & \xrightarrow{h_2} & B_3 & \xrightarrow{h_3} & B_4 & \xrightarrow{h_4} & B_5 \end{array}$$

where the horizontal sequences are exact. If f_1, f_2, f_4, f_5 are isomorphisms, then so is f_3.

Proof. First, we shall prove that f_3 is injective. It suffices to show that $\operatorname{Ker} f_3$ is trivial. Let $a_3 \in Ker f_3 \subseteq A_3$. Then $f_3(a_3) = 0$ gives us

$$h_3(f_3(a_3)) = h_3(0) = 0$$

Commutativity of the diagram gives us $h_3 \circ f_3 = f_4 \circ g_3$. So $f_4(g_3(a_3)) = 0$. Since f_4 is injective, we have $g_3(a_3) = 0$. In other words, $a_3 \in \operatorname{Ker} g_3 = \operatorname{im} g_2$. So $a_3 = g_2(a_2)$ for some $a_2 \in A_2$.

$$0 = f_3(a_3) = f_3(g_2(a_2)) = h_2(f_2(a_2))$$

Thus $f_2(a_2) \in \operatorname{Ker} h_2 = \operatorname{im} h_1$. So $f_2(a_2) = h_1(b_1)$ for some $b_1 \in B_1$. Since f_1 is surjective, $b_1 = f_1(a_1)$ for some $a_1 \in A_1$.

$$f_2(a_2) = h_1(b_1) = h_1(f_1(a_1)) = f_2(g_1(a_1))$$

Injectivity of f_1 gives us $g_1(a_1) = a_2$. Since $\operatorname{im} g_1 = \operatorname{Ker} g_2$, we have $g_2 \circ g_1 = 0$.

$$a_3 = g_2(a_2) = g_2(g_1(a_1)) = 0$$

Therefore, $\operatorname{Ker} f_3$ contains only the identity element, *i.e.* f_3 is injective.

Now let us prove that f_3 is surjective. Let $b_3 \in B_3$. We need to show that there exists some $a \in A_3$ such that $f_3(a) = b_3$.

$h_3(b_3) \in B_4$, and $f_4 : A_4 \to B_4$ is surjective. So $h_3(b_3) = f_4(a_4)$ for some $a_4 \in A_4$. Now, $h_4 \circ h_3 = 0$ as $\operatorname{im} h_3 = \operatorname{Ker} h_4$.

$$0 = h_4(h_3(b_3)) = h_4(f_4(a_4)) = f_5(g_4(a_4))$$

Using the injectivity of f_5, we get $g_4(a_4) = 0$. So $a_4 \in \operatorname{Ker} g_4 = \operatorname{im} g_3$. This gives us $a_4 = g_3(a_3)$ for some $a_3 \in A_3$. Now we claim that $b_3 - f_3(a_3) \in \operatorname{Ker} h_3$.

$$\begin{aligned} h_3(b_3 - f_3(a_3)) &= h_3(b_3) - h_3(f_3(a_3)) = f_4(a_4) - f_4(g_3(a_3)) \\ &= f_4(a_4) - f_4(a_4) = 0 \end{aligned}$$

Therefore, $b_3 - f_3(a_3) \in \operatorname{Ker} h_3 = \operatorname{im} h_2$. So $b_3 - f_3(a_3) = h_2(b_2)$ for some $b_2 \in B_2$. Since $f_2 : A_2 \to B_2$ is surjective, there exists $a_2 \in B_2$ such that $f_2(a_2) = b_2$.

Now take $a = g_2(a_2) + a_3$. Both $g_2(a_2)$ and a_3 are in A_3, so $a \in A_3$.

$$\begin{aligned} f_3(a) &= f_3(g_2(a_2) + a_3) = f_3(g_2(a_2)) + f_3(a_3) \\ &= h_2(f_2(a_2)) + f_3(a_3) = h_2(b_2) + f_3(a_3) \\ &= b_3 - f_3(a_3) + f_3(a_3) = b_3 \end{aligned}$$

So f_3 is surjective. Henceforth, f_3 is a bijective homomorphism, *i.e.* an isomorphism. ■

Lemma 5.1.3

Let $h : (K, K_0) \to (L, L_0)$ be a simplicial map ($h : K \to L$ is a simplicial map that takes each simplex of the subcomplex K_0 of K to a simplex of the subcomplex L of L_0).

(a) The induced homology homomorphisms $(h_*)_p : H_p(K, K_0) \to H_p(L, L_0)$ give a homomorphism of the exact homology sequences of (K, K_0) with that of (L, L_0).

(b) If $\left(\widetilde{h_*}\right)_i : H_i(K) \to H_i(L)$ and $\left(\widetilde{h_*^0}\right)_i : H_i(K_0) \to H_i(L_0)$ are isomorphism for $i = p$ and $i = p - 1$, then

$$(h_*)_p : H_p(K, K_0) \to H_p(L, L_0)$$

is an isomorphism.

(c) Both these results hold if absolute homology is replaced throughout by reduced homology.

Proof. First we prove (a). Recall from the previous chapter that the simplicial map $h:(K,K_0)\to(L,L_0)$ induces the map at the chain level:

$$(h_\#)_p : C_p(K,K_0)\to C_p(L,L_0)$$

One similarly obtains the following induced maps at the level of chains

$$\left(\widetilde{h_\#}\right)_p : C_p(K)\to C_p(L) \quad \text{and} \quad \left(\widetilde{h^0_\#}\right)_p : C_p(K_0)\to C_p(L_0)$$

As a matter of fact, there are the following exact sequences of chain complexes:

$$0\longrightarrow \mathcal{C}(K_0)\xrightarrow{i^K_\#}\mathcal{C}(K)\xrightarrow{\pi^K_\#}\mathcal{C}(K,K_0)\longrightarrow 0$$

$$0\longrightarrow \mathcal{C}(L_0)\xrightarrow{i^L_\#}\mathcal{C}(L)\xrightarrow{\pi^L_\#}\mathcal{C}(L,L_0)\longrightarrow 0$$

where the chain complex $\mathcal{C}(K)$, for example, is given by the family $\mathcal{C}(K)=\left\{C_p(K),\partial^K_p\right\}$. Then we have the following commutative diagram for every p.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_p(K_0) & \xrightarrow{(i^K_\#)_p} & C_p(K) & \xrightarrow{(\pi^K_\#)_p} & C_p(K,K_0) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \left(\widetilde{h^0_\#}\right)_p} & & \downarrow{\scriptstyle \left(\widetilde{h_\#}\right)_p} & & \downarrow{\scriptstyle (h_\#)_p} & & \\
0 & \longrightarrow & C_p(L_0) & \xrightarrow{(i^L_\#)_p} & C_p(L) & \xrightarrow{(\pi^L_\#)_p} & C_p(L,L_0) & \longrightarrow & 0
\end{array}$$

Each of the maps $\widetilde{h^0_\#}:\mathcal{C}(K_0)\to\mathcal{C}(L_0)$, $\widetilde{h_\#}:\mathcal{C}(K)\to\mathcal{C}(L)$ and $h_\#:\mathcal{C}(K,K_0)\to\mathcal{C}(L,L_0)$ are chain maps between the respective chain complexes.

$$\mathcal{C}(K,K_0)\equiv\left\{C_p(K,K_0),\partial_p^{(K,K_0)}\right\}\ ,\ \mathcal{C}(K_0)\equiv\left\{C_p(K_0),\partial_p^{K_0}\right\}$$

One can easily check that $(h_\#)_p : C_p(K,K_0)\to C_p(L,L_0)$ commutes with the relative boundary operator, *i.e.* $\partial_p^{(L,L_0)}\circ(h_\#)_p=(h_\#)_{p-1}\circ\partial_p^{(K,K_0)}$. In other words, the following diagram commutes:

$$\begin{array}{ccc}
C_p(K,K_0) & \xrightarrow{(h_\#)_p} & C_p(L,L_0)\\
\downarrow{\scriptstyle \partial_p^{(K,K_0)}} & & \downarrow{\scriptstyle \partial_p^{(L,L_0)}}\\
C_{p-1}(K,K_0) & \xrightarrow{(h_\#)_{p-1}} & C_{p-1}(L,L_0)
\end{array}$$

Hence, using our knowledge from previous chapter, there is an induced homomorphim

$$(h_*)_p : H_p(K,K_0)\to H_p(L,L_0)\ .$$

Similarly, there are other induced homorphisms

$$\left(\widetilde{h_*}\right)_p : H_p(K)\to H_p(L) \quad \text{and} \quad \left(\widetilde{h^0_*}\right)_p : H_p(K_0)\to H_p(L_0)\ .$$

By means of Theorem 4.3.1, since

$$0\longrightarrow \mathcal{C}(K_0)\xrightarrow{i^K_\#}\mathcal{C}(K)\xrightarrow{\pi^K_\#}\mathcal{C}(K,K_0)\longrightarrow 0$$

is exact sequence of chain complexes, there is a long exact homology sequence:

$$\cdots\longrightarrow H_p(K_0)\xrightarrow{(i^K_*)_p}H_p(K)\xrightarrow{(\pi^K_*)_p}H_p(K,K_0)\xrightarrow{(\partial^K_*)_p}H_{p-1}(K_0)\longrightarrow\cdots$$

Similarly, there is a long exact sequence for L and L_0:

$$\cdots \longrightarrow H_p(L_0) \xrightarrow{\left(i_*^L\right)_p} H_p(L) \xrightarrow{\left(\pi_*^L\right)_p} H_p(L, L_0) \xrightarrow{\left(\partial_*^L\right)_p} H_{p-1}(L_0) \longrightarrow \cdots$$

These two long exact sequences and the homomorphism given by the family

$$\left\{\ldots, \left(\widetilde{h_*^0}\right)_p, \left(\widetilde{h_*}\right)_p, (h_*)_p, \left(\widetilde{h_*^0}\right)_{p-1}, \ldots\right\}$$

between the two exact homology sequences give the following commutative diagram:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_p(K_0) & \xrightarrow{\left(i_*^K\right)_p} & H_p(K) & \xrightarrow{\left(\pi_*^K\right)_p} & H_p(K, K_0) & \xrightarrow{\left(\partial_*^K\right)_p} & H_{p-1}(K_0) & \longrightarrow \cdots \\
& \Big\downarrow {\scriptstyle \left(\widetilde{h_*^0}\right)_p} & & \Big\downarrow {\scriptstyle \left(\widetilde{h_*}\right)_p} & & \Big\downarrow {\scriptstyle (h_*)_p} & & \Big\downarrow {\scriptstyle \left(\widetilde{h_*^0}\right)_{p-1}} & \\
\cdots \longrightarrow & H_p(L_0) & \xrightarrow[\left(i_*^L\right)_p]{} & H_p(L) & \xrightarrow[\left(\pi_*^L\right)_p]{} & H_p(L, L_0) & \xrightarrow[\left(\partial_*^L\right)_p]{} & H_{p-1}(L_0) & \longrightarrow \cdots
\end{array}$$

Commutativity of the diagram follows from Theorem 5.1.1.

Now we are gonna show that this holds for reduced homology. We have the following non-negative chain complexes $\mathcal{C}(K) \equiv \left\{C_p(K), \partial_p^K\right\}$, $\mathcal{C}(K_0) \equiv \left\{C_p(K_0), \partial_p^{K_0}\right\}$; and the chain map $i_\#^K : \mathcal{C}(K_0) \to \mathcal{C}(K)$. Using the definition of ϵ, one can show that $i_\#^K$ preserves augmentation. In other words, the following diagram commutes:

$$\begin{array}{ccc}
C_0(K_0) & \xrightarrow{\left(i_\#^K\right)_0} & C_0(K) \\
\Big\downarrow {\scriptstyle \epsilon} & & \Big\downarrow {\scriptstyle \epsilon} \\
\mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z}
\end{array}$$

Therefore, $i_\#^K$ can be seen to be a chain map between the augmented chain complexes $\{\mathcal{C}(K_0), \epsilon\}$ and $\{\mathcal{C}(K), \epsilon\}$. Now since $C_{-1}(K) = C_{-1}(K_0) = \mathbb{Z}$, we have $C_{-1}(K, K_0) = 0$. So the following diagram commutes:

$$\begin{array}{ccc}
C_0(K) & \xrightarrow{\left(\pi_\#^K\right)_0} & C_0(K, K_0) \\
\Big\downarrow {\scriptstyle \epsilon} & & \Big\downarrow {\scriptstyle \overline{0}} \\
\mathbb{Z} & \xrightarrow{\overline{0}} & 0
\end{array}$$

(The 0-map is denoted by $\overline{0}$ which maps everything to 0.) So one can conclude that $\pi_\#^K$ is a chain map between the augmented chain complexes $\{\mathcal{C}(K), \epsilon\}$ and $\left\{\mathcal{C}(K, K_0), \overline{0}\right\}$. For notational convenience, we denote the chain map between augmented chain complexes with the same symbol used to denote the chain map between the respective chain complexes. For example:

$$\pi_\#^K : \mathcal{C}(K) \to \mathcal{C}(K, K_0) \quad \text{and} \quad \pi_\#^K : \{\mathcal{C}(K), \epsilon\} \to \left\{\mathcal{C}(K, K_0), \overline{0}\right\},$$

so that one has the following commutative diagram of augmented chain complexes:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \left\{C_p(K_0), \partial_p^{K_0}, \epsilon\right\} & \xrightarrow{\left(i_\#^K\right)_p} & \left\{C_p(K), \partial_p^K, \epsilon\right\} & \xrightarrow{\left(\pi_\#^K\right)_p} & \left\{C_p(K, K_0), \partial_p^{(K,K_0)}, \overline{0}\right\} & \longrightarrow 0 \\
& \Big\downarrow {\scriptstyle \left(\widetilde{h_\#^0}\right)_p} & & \Big\downarrow {\scriptstyle \left(\widetilde{h_\#}\right)_p} & & \Big\downarrow {\scriptstyle (h_\#)_p} & \\
0 \longrightarrow & \left\{C_p(L_0), \partial_p^{L_0}, \epsilon\right\} & \xrightarrow{\left(i_\#^L\right)_p} & \left\{C_p(L), \partial_p^L, \epsilon\right\} & \xrightarrow{\left(\pi_\#^L\right)_p} & \left\{C_p(L, L_0), \partial_p^{(L,L_0)}, \overline{0}\right\} & \longrightarrow 0
\end{array}$$

Now we repeat the same arguments presented in the beginning of the proof for absolute homology groups. There are induced homomorphisms between the respective homology groups of the augmented chain complexes. For instance,

$$\left(i_*^K\right)_p : H_p\left(\left\{\mathcal{C}\left(K_0\right), \epsilon\right\}\right) \to H_p\left(\left\{\mathcal{C}\left(K\right), \epsilon\right\}\right)$$

is the induced homomorphisms between the reduced homology groups of the respective chain complexes:

$$\left(i_*^K\right)_p : \widetilde{H}_p\left(K_0\right) \to \widetilde{H}_p\left(K\right)$$

Using the same arguments presented in the beginning of the proof for absolute homology groups, one obtains the following commutative diagram:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \widetilde{H}_p\left(K_0\right) & \xrightarrow{\left(i_*^K\right)_p} & \widetilde{H}_p\left(K\right) & \xrightarrow{\left(\pi_*^K\right)_p} & \widetilde{H}_p\left(K, K_0\right) & \xrightarrow{\left(\partial_*^K\right)_p} & \widetilde{H}_{p-1}\left(K_0\right) & \longrightarrow \cdots \\
& \Big\downarrow {\scriptstyle \left(\widetilde{h_*^0}\right)_p} & & \Big\downarrow {\scriptstyle \left(\widetilde{h_*}\right)_p} & & \Big\downarrow {\scriptstyle \left(h_*\right)_p} & & \Big\downarrow {\scriptstyle \left(\widetilde{h_*^0}\right)_{p-1}} & \\
\cdots \longrightarrow & \widetilde{H}_p\left(L_0\right) & \xrightarrow[\left(i_*^L\right)_p]{} & \widetilde{H}_p\left(L\right) & \xrightarrow[\left(\pi_*^L\right)_p]{} & \widetilde{H}_p\left(L, L_0\right) & \xrightarrow[\left(\partial_*^L\right)_p]{} & \widetilde{H}_{p-1}\left(L_0\right) & \longrightarrow \cdots
\end{array}$$

Commutativity of the diagram follows similarly from Theorem 5.1.1.

Now we shall prove (b). Consider the following commutative diagram.

$$\begin{array}{ccccccccc}
H_p\left(K_0\right) & \xrightarrow{\left(i_*^K\right)_p} & H_p\left(K\right) & \xrightarrow{\left(\pi_*^K\right)_p} & H_p\left(K, K_0\right) & \xrightarrow{\left(\partial_*^K\right)_p} & H_{p-1}\left(K_0\right) & \xrightarrow{\left(i_*^K\right)_{p-1}} & H_{p-1}\left(K\right) \\
\Big\downarrow {\scriptstyle \left(\widetilde{h_*^0}\right)_p} & & \Big\downarrow {\scriptstyle \left(\widetilde{h_*}\right)_p} & & \Big\downarrow {\scriptstyle \left(h_*\right)_p} & & \Big\downarrow {\scriptstyle \left(\widetilde{h_*^0}\right)_{p-1}} & & \Big\downarrow {\scriptstyle \left(\widetilde{h_*}\right)_{p-1}} \\
H_p\left(L_0\right) & \xrightarrow[\left(i_*^L\right)_p]{} & H_p\left(L\right) & \xrightarrow[\left(\pi_*^L\right)_p]{} & H_p\left(L, L_0\right) & \xrightarrow[\left(\partial_*^L\right)_p]{} & H_{p-1}\left(L_0\right) & \xrightarrow{\left(i_*^L\right)_{p-1}} & H_{p-1}\left(L\right)
\end{array}$$

We are given that the first two and the last two vertical maps are isomorphisms. Therefore, by Steenrod Five Lemma, $\left(h_*\right)_p : H_p\left(K, K_0\right) \to H_p\left(L, L_0\right)$ is an isomorphism.

Similarly, for augmented chain complexes:

$$\begin{array}{ccccccccc}
\widetilde{H}_p\left(K_0\right) & \xrightarrow{\left(i_*^K\right)_p} & \widetilde{H}_p\left(K\right) & \xrightarrow{\left(\pi_*^K\right)_p} & \widetilde{H}_p\left(K, K_0\right) & \xrightarrow{\left(\partial_*^K\right)_p} & \widetilde{H}_{p-1}\left(K_0\right) & \xrightarrow{\left(i_*^K\right)_{p-1}} & \widetilde{H}_{p-1}\left(K\right) \\
\Big\downarrow {\scriptstyle \left(\widetilde{h_*^0}\right)_p} & & \Big\downarrow {\scriptstyle \left(\widetilde{h_*}\right)_p} & & \Big\downarrow {\scriptstyle \left(h_*\right)_p} & & \Big\downarrow {\scriptstyle \left(\widetilde{h_*^0}\right)_{p-1}} & & \Big\downarrow {\scriptstyle \left(\widetilde{h_*}\right)_{p-1}} \\
\widetilde{H}_p\left(L_0\right) & \xrightarrow[\left(i_*^L\right)_p]{} & \widetilde{H}_p\left(L\right) & \xrightarrow[\left(\pi_*^L\right)_p]{} & \widetilde{H}_p\left(L, L_0\right) & \xrightarrow[\left(\partial_*^L\right)_p]{} & \widetilde{H}_{p-1}\left(L_0\right) & \xrightarrow{\left(i_*^L\right)_{p-1}} & \widetilde{H}_{p-1}\left(L\right)
\end{array}$$

Using Steenrod Five Lemma again, oone finds that $\left(h_*\right)_p : \widetilde{H}_p\left(K, K_0\right) \to \widetilde{H}_p\left(L, L_0\right)$ is an isomorphism whenever

$$\left(\widetilde{h_*}\right)_i : \widetilde{H}_i\left(K\right) \to \widetilde{H}_i\left(L\right) \quad \text{and} \quad \left(\widetilde{h_*^0}\right)_i : \widetilde{H}_i\left(K_0\right) \to \widetilde{H}_i\left(L_0\right)$$

are isomorphisms between the respective reduced homology groups for $i = p, p-1$. ■

§5.2 Mayer-Vietoris Sequences

Theorem 5.2.1

Let K be a complex; let K_0 and K_1 be subcomplexes such that $K = K_0 \cup K_1$. Let $A = K_0 \cap K_1$. Then there is an exact sequence

$$\cdots \longrightarrow H_p\left(A\right) \longrightarrow H_p\left(K_0\right) \oplus H_p\left(K_1\right) \longrightarrow H_p\left(K\right) \longrightarrow H_{p-1}\left(A\right) \longrightarrow \cdots,$$

called the **Mayer-Vietoris sequence** of (K_0, K_1). There is a similar exact sequence in reduced homology if A is nonempty.

Proof. The proof consists of constructing a short exact sequence of chain complexes

$$0 \longrightarrow \mathcal{C}(A) \xrightarrow{\phi} \mathcal{C}(K_0) \oplus \mathcal{C}(K_1) \xrightarrow{\psi} \mathcal{C}(K) \longrightarrow 0$$

and applying The Zig-Zag Lemma.

First of all, we need to define the chain complex $\mathcal{C}(K_0) \oplus \mathcal{C}(K_1)$. Its chain group in dimension p is defined as $C_p(K_0) \oplus C_p(K_1)$; and its boundary operator ∂'_p is defined as

$$\partial'_p(d_p, e_p) = \left(\partial_p^{K_0} d_p, \partial_p^{K_1} e_p\right)$$

with $\partial_p^{K_0}$ and $\partial_p^{K_1}$ being boundary operators in $\mathcal{C}(K_0)$ and $\mathcal{C}(K_1)$, respectively. Clearly, this satisfies $\partial'_p \circ \partial'_{p+1} = 0$.

Secondly, we need to define the chain maps ϕ and ψ. Consider the inclusion mappings in the following commutative diagram:

$$\begin{array}{ccccc}
 & & K_0 & & \\
 & {\scriptstyle i}\nearrow & & \searrow{\scriptstyle k} & \\
A = K_0 \cap K_1 & & \xrightarrow{\quad m \quad} & & K = K_0 \cup K_1 \\
 & {\scriptstyle j}\searrow & & \nearrow{\scriptstyle l} & \\
 & & K_1 & &
\end{array}$$

i, j, k, l induce four chain maps $i_\#, j_\#, k_\#, l_\#$:

$$(i_\#)_p : C_p(A) \to C_p(K_0) \ , \ (j_\#)_p : C_p(A) \to C_p(K_1)$$

$$(k_\#)_p : C_p(K_0) \to C_p(K) \ , \ (l_\#)_p : C_p(K_1) \to C_p(K)$$

$$\begin{array}{ccccc}
 & & C_p(K_0) & & \\
 & {\scriptstyle (i_\#)_p}\nearrow & & \searrow{\scriptstyle (k_\#)_p} & \\
C_p(A) & & \xrightarrow{\quad (m_\#)_p \quad} & & C_p(K) \\
 & {\scriptstyle (j_\#)_p}\searrow & & \nearrow{\scriptstyle (l_\#)_p} & \\
 & & C_p(K_1) & &
\end{array}$$

We define ϕ and ψ as follows: Let $c_p \in C_p(A)$. Then we define $\phi_p : C_p(A) \to C_p(K_0) \oplus C_p(K_1)$ as

$$\phi_p(c_p) = \Big((i_\#)_p c_p, -(j_\#)_p c_p\Big) .$$

Now let $(d_p, e_p) \in C_p(K_0) \oplus C_p(K_1)$, where $d_p \in C_p(K_0)$ and $e_p \in C_p(K_1)$. We define $\psi_p : C_p(K_0) \oplus C_p(K_1) \to C_p(K)$ as:

$$\psi_p(d_p, e_p) = (k_\#)_p d_p + (l_\#)_p e_p .$$

To show that ϕ is a chain map, we need to verify that the following diagram commutes:

$$\begin{array}{ccc}
C_{p+1}(A) & \xrightarrow{\quad \partial_{p+1}^A \quad} & C_p(A) \\
\Big\downarrow{\scriptstyle \phi_{p+1}} & & \Big\downarrow{\scriptstyle \phi_p} \\
C_{p+1}(K_0) \oplus C_{p+1}(K_1) & \xrightarrow{\partial'_{p+1}} & C_p(K_0) \oplus C_p(K_1)
\end{array}$$

Since both $i_\#$ and $j_\#$ are chain maps, both of the following diagrams commute:

$$\begin{array}{ccc} C_{p+1}(A) & \xrightarrow{(i_\#)_{p+1}} & C_{p+1}(K_0) \\ \downarrow{\scriptstyle \partial^A_{p+1}} & & \downarrow{\scriptstyle \partial^{K_0}_{p+1}} \\ C_p(A) & \xrightarrow{(i_\#)_p} & C_p(K_0) \end{array} \qquad \begin{array}{ccc} C_{p+1}(A) & \xrightarrow{(j_\#)_{p+1}} & C_{p+1}(K_1) \\ \downarrow{\scriptstyle \partial^A_{p+1}} & & \downarrow{\scriptstyle \partial^{K_1}_{p+1}} \\ C_p(A) & \xrightarrow{(j_\#)_p} & C_p(K_1) \end{array}$$

Therefore, for a given $c_{p+1} \in C_{p+1}(A)$,

$$\begin{aligned} \phi_p\left(\partial^A_{p+1} c_{p+1}\right) &= \left((i_\#)_p\, \partial^A_{p+1} c_{p+1}, -(j_\#)_p\, \partial^A_{p+1} c_{p+1}\right) \\ &= \left(\partial^{K_0}_{p+1}\left((i_\#)_{p+1} c_{p+1}\right), -\partial^{K_1}_{p+1}\left((j_\#)_{p+1} c_{p+1}\right)\right) \\ &= \partial'_{p+1}\left((i_\#)_{p+1} c_{p+1}, -(j_\#)_{p+1} c_{p+1}\right) \\ &= \partial'_{p+1}\left(\phi_{p+1}(c_{p+1})\right) \end{aligned}$$

Hence, $\phi_p \circ \partial^A_{p+1} = \partial'_{p+1} \circ \phi_{p+1}$; so ϕ is a chain map.

Now we are gonna check that ψ is also a chain map by showing that the following diagram commutes.

$$\begin{array}{ccc} C_{p+1}(K_0) \oplus C_{p+1}(K_1) & \xrightarrow{\partial'_{p+1}} & C_p(K_0) \oplus C_p(K_1) \\ \downarrow{\scriptstyle \psi_{p+1}} & & \downarrow{\scriptstyle \psi_p} \\ C_{p+1}(K) & \xrightarrow{\partial^K_{p+1}} & C_p(K) \end{array}$$

Since both $(k_\#)_p$ and $(l_\#)_p$ are chain maps, both of the following diagrams commute:

$$\begin{array}{ccc} C_{p+1}(K_0) & \xrightarrow{(k_\#)_{p+1}} & C_{p+1}(K) \\ \downarrow{\scriptstyle \partial^{K_0}_{p+1}} & & \downarrow{\scriptstyle \partial^{K}_{p+1}} \\ C_p(K_0) & \xrightarrow{(k_\#)_p} & C_p(K) \end{array} \qquad \begin{array}{ccc} C_{p+1}(K_1) & \xrightarrow{(l_\#)_{p+1}} & C_{p+1}(K) \\ \downarrow{\scriptstyle \partial^{K_1}_{p+1}} & & \downarrow{\scriptstyle \partial^{K}_{p+1}} \\ C_p(K_1) & \xrightarrow{(l_\#)_p} & C_p(K) \end{array}$$

Now let $(d_{p+1}, e_{p+1}) \in C_{p+1}(K_0) \oplus C_{p+1}(K_1)$, where $d_{p+1} \in C_{p+1}(K_0)$ and $e_{p+1} \in C_{p+1}(K_1)$.

$$\begin{aligned} \psi_p\left(\partial'_{p+1}(d_{p+1}, e_{p+1})\right) &= \psi_p\left(\partial^{K_0}_{p+1} d_{p+1}, \partial^{K_1}_{p+1} e_{p+1}\right) \\ &= (k_\#)_p\left(\partial^{K_0}_{p+1} d_{p+1}\right) + (l_\#)_p\left(\partial^{K_1}_{p+1} e_{p+1}\right) \\ &= \partial^K_{p+1}\left((k_\#)_{p+1} d_{p+1}\right) + \partial^K_{p+1}\left((l_\#)_{p+1} e_{p+1}\right) \\ &= \partial^K_{p+1}\left[\left((k_\#)_{p+1} d_{p+1}\right) + (l_\#)_{p+1} e_{p+1}\right] \\ &= \partial^K_{p+1}\left(\psi_{p+1}(d_{p+1}, e_{p+1})\right) \end{aligned}$$

Hence, $\psi_p \circ \partial'_{p+1} = \partial^K_{p+1} \circ \psi_{p+1}$; so ψ is a chain map.

Let us now check the exactness of the sequence

$$0 \longrightarrow C_p(A) \xrightarrow{\phi_p} C_p(K_0) \oplus C_p(K_1) \xrightarrow{\psi_p} C_p(K) \longrightarrow 0$$

Since $(i_\#)_p$ and $(j_\#)_p$ are just inclusion of p-chains, from the definition of ϕ,

$$\phi_p(c_p) = \left((i_\#)_p\, c_p, -(j_\#)_p\, c_p\right),$$

one finds that ϕ_p is injective for every p.

Let us now check that ψ_p is surjective for every p. Given $d \in C_p(K)$, we write d as an integral linear combination of elementary p-chains corresponding to oriented p-simplices. Let d_0 consist of those terms carried by K_0. Then $d - d_0$ is carried by K_1 since $K = K_0 \cup K_1$. Then we get

$$\psi_p(d_0, d - d_0) = (k_\#)_p(d_0) + (l_\#)_p(d - d_0) = d_0 + d - d_0 = d$$

Thus ψ_p is surjective for every p.

Now we shall check exactness at $C_p(K_0) \oplus C_p(K_1)$. For that we need to show that $\operatorname{im}\phi_p = \operatorname{Ker}\psi_p$. Let $c_p \in C_p(A)$.

$$\begin{aligned}\psi_p(\phi_p(c_p)) &= \psi_p\Big((i_\#)_p c_p, -(j_\#)_p c_p\Big) = (k_\#)_p\Big((i_\#)_p c_p\Big) - (l_\#)_p\Big((j_\#)_p c_p\Big)\\ &= (m_\#)_p(c_p) - (m_\#)_p(c_p) = 0\end{aligned}$$

So $\operatorname{im}\phi_p \subseteq \operatorname{Ker}\psi_p$. Conversely, suppose $\psi_p(d_p, e_p) = 0$ for $d_p \in C_p(K_0)$ and $e_p \in C_p(K_1)$.

$$0 = \psi_p(d_p, e_p) = (k_\#)_p d_p + (l_\#)_p e_p = d_p + e_p\,,$$

because $(k_\#)_p$ and $(l_\#)_p$ are inclusions. So we have $d_p = -e_p$. But d_p is carried by K_0 and e_p is carried by K_1. So they must be both carried by $K_0 \cap K_1 = A$. In other words, $d_p = -e_p \in C_p(A)$. Therefore, $d_p = (i_\#)_p d_p$ and $d_p = (j_\#)_p d_p$.

$$(d_p, e_p) = (d_p, -d_p) = \Big((i_\#)_p d_p, -(j_\#)_p d_p\Big) = \phi_p(d_p) \in \operatorname{im}\phi_p$$

Thus $\operatorname{Ker}\psi_p \subseteq \operatorname{im}\phi_p$. Henceforth, $\operatorname{im}\phi_p = \operatorname{Ker}\psi_p$, and the sequence

$$0 \longrightarrow C_p(A) \xrightarrow{\phi_p} C_p(K_0) \oplus C_p(K_1) \xrightarrow{\psi_p} C_p(K) \longrightarrow 0$$

is exact.

Now we shall compute the homology groups of the chain complex $\mathcal{C}(K_0) \oplus \mathcal{C}(K_1)$ in dimension p. One can easily verify that $\operatorname{Ker}\partial'_p = \operatorname{Ker}\partial_p^{K_0} \oplus \operatorname{Ker}\partial_p^{K_1}$, and $\operatorname{im}\partial'_{p+1} = \operatorname{im}\partial_{p+1}^{K_0} \oplus \operatorname{im}\partial_{p+1}^{K_1}$. Therefore,

$$H_p(\mathcal{C}(K_0) \oplus \mathcal{C}(K_1)) = \operatorname{Ker}\partial'_p / \operatorname{im}\partial'_{p+1} = \frac{\operatorname{Ker}\partial_p^{K_0} \oplus \operatorname{Ker}\partial_p^{K_1}}{\operatorname{im}\partial_{p+1}^{K_0} \oplus \operatorname{im}\partial_{p+1}^{K_1}}$$

Recall from the abelian groups essentials that we studied in *Algebraic Topology I* that

$$\frac{G_1 \oplus G_2}{H_1 \oplus H_2} \cong \frac{G_1}{H_1} \oplus \frac{G_2}{H_2}\ , \quad \text{if } H_i \text{ is a subgroup of } G_i \text{ for } i = 1, 2\,.$$

$\operatorname{im}\partial_{p+1}^{K_i}$ is a subgroup of $\operatorname{Ker}\partial_p^{K_i}$ for $i = 0, 1$. Therefore, we have

$$H_p(\mathcal{C}(K_0) \oplus \mathcal{C}(K_1)) = \frac{\operatorname{Ker}\partial_p^{K_0} \oplus \operatorname{Ker}\partial_p^{K_1}}{\operatorname{im}\partial_{p+1}^{K_0} \oplus \operatorname{im}\partial_{p+1}^{K_1}} \cong \frac{\operatorname{Ker}\partial_p^{K_0}}{\operatorname{im}\partial_{p+1}^{K_0}} \oplus \frac{\operatorname{Ker}\partial_p^{K_1}}{\operatorname{im}\partial_{p+1}^{K_1}} = H_p(K_0) \oplus H_p(K_1)$$

To apply The Zig-Zag Lemma, let $\mathcal{C} = \mathcal{C}(K_0 \cap K_1) = \mathcal{C}(A)$, $\mathcal{D} = \mathcal{C}(K_0) \oplus \mathcal{C}(K_1)$, and $\mathcal{E} = \mathcal{C}(K_0 \cup K_1) = \mathcal{C}(K)$. Then $H_p(\mathcal{C}) = H_p(A)$, $H_p(\mathcal{D}) \cong H_p(K_0) \oplus H_p(K_1)$, $H_p(\mathcal{E}) = H_p(K)$. After applying The Zig-Zag Lemma, we get the following exact homology sequence:

$$\begin{array}{ccccc} & & \cdots & \longrightarrow & H_{p+1}(K) \\ & \swarrow & & & \\ H_p(A) & \longrightarrow & H_p(K_0) \oplus H_p(K_1) & \longrightarrow & H_p(K) \\ & \swarrow & & & \\ H_{p-1}(A) & \longrightarrow & \cdots & & \end{array}$$

To obtain the Mayer-Vietoris sequence in reduced homology, let $\epsilon_A, \epsilon_0, \epsilon_1, \epsilon$ denote the augmentation maps for $\mathcal{C}(A), \mathcal{C}(K_0), \mathcal{C}(K_1), \mathcal{C}(K)$ respectively.

We want to show that

$$\phi : \{\mathcal{C}(A), \epsilon_A\} \to \{\mathcal{C}(K_0) \oplus \mathcal{C}(K_1), \epsilon_0 \oplus \epsilon_1\} \text{ and } \psi : \{\mathcal{C}(K_0) \oplus \mathcal{C}(K_1), \epsilon_0 \oplus \epsilon_1\} \to \{\mathcal{C}(K), \epsilon\}$$

are chain maps between the respective augmented chain complexes. Let's define $\phi_{-1} : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ by $\phi_{-1}(n) = (n, -n)$ and $\psi_{-1} : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ by $\phi_{-1}(m, n) = m + n$. We need to show that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(A) & \xrightarrow{\phi_0} & C_0(K_0) \oplus C_0(K_1) & \xrightarrow{\psi_0} & C_0(K) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \epsilon_A} & & \downarrow{\scriptstyle \epsilon_0 \oplus \epsilon_1} & & \downarrow{\scriptstyle \epsilon} & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\phi_{-1}} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\psi_{-1}} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

First we want to prove that the square on the left of the diagram commutes, *i.e.* $(\epsilon_0 \oplus \epsilon_1) \circ \phi_0 = \phi_{-1} \circ \epsilon_A$.

Both the chain maps $i_\#$ and $j_\#$ are easily seen to be augmentation preserving. In other words, the following diagrams commute:

$$\begin{array}{ccc} C_0(A) & \xrightarrow{(i_\#)_0} & C_0(K_0) \\ \downarrow{\scriptstyle \epsilon_A} & & \downarrow{\scriptstyle \epsilon_0} \\ \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \end{array} \qquad \begin{array}{ccc} C_0(A) & \xrightarrow{(j_\#)_0} & C_0(K_1) \\ \downarrow{\scriptstyle \epsilon_A} & & \downarrow{\scriptstyle \epsilon_1} \\ \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \end{array}$$

Now, for a given 0-chain $c_0 \in C_0(A)$, we have

$$\begin{aligned} ((\epsilon_0 \oplus \epsilon_1) \circ \phi_0)(c_0) &= (\epsilon_0 \oplus \epsilon_1)\left((i_\#)_0 c_0, -(j_\#)_0 c_0\right) = \left(\epsilon_0\left((i_\#)_0 c_0\right), \epsilon_1\left((j_\#)_0 c_0\right)\right) \\ &= (\epsilon_A c_0, -\epsilon_A c_0) = \phi_{-1}(\epsilon_A c_0) = (\phi_{-1} \circ \epsilon_A)(c_0) \end{aligned}$$

So we have proved $(\epsilon_0 \oplus \epsilon_1) \circ \phi_0 = \phi_{-1} \circ \epsilon_A$. Now we want to prove that $\epsilon \circ \psi_0 = \psi_{-1} \circ (\epsilon_0 \oplus \epsilon_1)$.

Similar as before, $k_\#$ and $l_\#$ are easily seen to be augmentation preserving. In other words, the following diagrams commute:

$$\begin{array}{ccc} C_0(K_0) & \xrightarrow{(k_\#)_0} & C_0(K) \\ \downarrow{\scriptstyle \epsilon_0} & & \downarrow{\scriptstyle \epsilon} \\ \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \end{array} \qquad \begin{array}{ccc} C_0(K_1) & \xrightarrow{(l_\#)_0} & C_0(K) \\ \downarrow{\scriptstyle \epsilon_1} & & \downarrow{\scriptstyle \epsilon} \\ \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \end{array}$$

Let $(c_0, c_1) \in C_0(K_0) \oplus C_0(K_1)$ with $c_0 \in C_0(K_0)$ and $c_1 \in C_0(K_1)$.

$$\begin{aligned} (\epsilon \circ \psi_0)(c_0, c_1) &= \epsilon\left((k_\#)_0 c_0 + (l_\#)_0 c_1\right) = \epsilon\left((k_\#)_0 c_0\right) + \epsilon\left((l_\#)_0 c_1\right) \\ &= \epsilon_0 c_0 + \epsilon_1 c_1 = \psi_{-1}(\epsilon_0 c_0, \epsilon_1 c_1) \\ &= (\psi_{-1} \circ (\epsilon_0 \oplus \epsilon_1))(c_0, c_1) \end{aligned}$$

Therefore, $\epsilon \circ \psi_0 = \psi_{-1} \circ (\epsilon_0 \oplus \epsilon_1)$. Hence, ϕ and ψ are chain maps between the respective augmented chain complexes. Furthermore,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\phi_{-1}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\psi_{-1}} \mathbb{Z} \longrightarrow 0$$

is a short exact sequence: clearly, ϕ_{-1} is injective and ψ_{-1} is surjective. Also,

$$\begin{aligned} \operatorname{Ker} \psi_{-1} &= \{(m, n) \in \mathbb{Z} \oplus \mathbb{Z} \ : \ m + n = 0\} \\ &= \{(n, -n) \ : \ n \in \mathbb{Z}\} = \operatorname{im} \phi_{-1} \end{aligned}$$

Therefore, one has the following short exact sequence of augmented chain complexes:

$$0 \longrightarrow \{\mathcal{C}(A), \epsilon_A\} \xrightarrow{\phi} \{\mathcal{C}(K_0) \oplus \mathcal{C}(K_1), \epsilon_0 \oplus \epsilon_1\} \xrightarrow{\psi} \{\mathcal{C}(K), \epsilon\} \longrightarrow 0$$

Then applying The Zig-Zag Lemma, one obtains the Mayer-Vietoris sequence in reduced homology:

$$\cdots \longrightarrow \widetilde{H_{p+1}}(K)$$
$$\widetilde{H_p}(A) \longrightarrow \widetilde{H_p}(K_0) \oplus \widetilde{H_p}(K_1) \longrightarrow \widetilde{H_p}(K)$$
$$\widetilde{H_{p-1}}(A) \longrightarrow \cdots$$

■

Fact. Let $0 \longrightarrow A \xrightarrow{\phi} B \longrightarrow 0$ be an exact sequence of abelian groups. Then exactness at A gives us that $\operatorname{Ker}\phi = 0$, which means ϕ is an injective homomorphism. Exactness at B yields $\operatorname{im}\phi = B$, so ϕ is surjective. Therefore, ϕ is a bijective homomrphism, *i.e.* an isomorphism.

Definition 5.2.1 (Suspension). Let K be a complex; let $w_0 * K$ and $w_1 * K$ be two cones on K whose polytopes intersect in $|K|$ alone. Then

$$S(K) = (w_0 * K) \cup (w_1 * K)$$

is a complex; it is called a **suspension** of K. Given K, the complex $S(K)$ is defined uniquely up to a simplicial isomorphism.

Theorem 5.2.2

If K is a complex, then for all p, there is an isomorphism

$$\widetilde{H_p}\left(S\left(K\right)\right) \to \widetilde{H_{p-1}}\left(K\right) .$$

Proof. Let $K_0 = w_0 * K$ and $K_1 = w_1 * K$. Then $K_0 \cup K_1 = S\left(K\right)$, and $K_0 \cap K_1 = K$ is nonempty. Therefore, Mayer-Vietoris sequence in reduced homology yields the following exact sequence:

$$\widetilde{H_p}\left(K_0\right) \oplus \widetilde{H_p}\left(K_1\right) \to \widetilde{H_p}\left(S\left(K\right)\right) \to \widetilde{H_{p-1}}\left(K\right) \to \widetilde{H_{p-1}}\left(K_0\right) \oplus \widetilde{H_{p-1}}\left(K_1\right)$$

We know from Theorem 2.3.4 that a cone is acyclic. In other words, $\widetilde{H_p}\left(K_0\right) = 0$ and $\widetilde{H_p}\left(K_1\right) = 0$ for every p. Therefore, the exact sequence stated above becomes

$$0 \to \widetilde{H_p}\left(S\left(K\right)\right) \to \widetilde{H_{p-1}}\left(K\right) \to 0$$

Then using this fact, $\widetilde{H_p}\left(S\left(K\right)\right) \to \widetilde{H_{p-1}}\left(K\right)$ is an isomorphism for every p. ■

6 Lecture 6

§6.1 Eilenberg-Steenrod Axioms

Definition 6.1.1 (Admissible Class)**.** Let $\mathcal{A}$ be a class of pairs (X, A) of topological spaces, with A being a subspace of X, such that

1. If (X, A) belong to the class $\mathcal{A}$, then so do (X, X), $(X, \varnothing)$, (A, A) and $(A, \varnothing)$.

2. If (X, A) belong to the class $\mathcal{A}$, then so does $(X \times I, A \times I)$, where I is the unit interval $[0, 1]$.[a]

3. There is a one-point space P such that $(P, \varnothing)$ is in $\mathcal{A}$.

We shall call $\mathcal{A}$ an **admissible class of spaces** for a homology theory.

[a] It's immediate that given topological space X and its subspace A, then $X \times I$ and $A \times I$ are topological spaces in product topology; and $A \times I$ is a subspace of $X \times I$.

Definition 6.1.2. If $\mathcal{A}$ is admissible, then a **homology theory** on $\mathcal{A}$ consists of three functions:

1. A function H_p defined for each pair in the admissible class $\mathcal{A}$; $(X, A) \mapsto H_p(X, A)$ where $H_p(X, A)$ is an abelian group. H_p is defined for each integer p.

2. A function that, for each integer p, assigns to each continuous map $h : (X, A) \to (Y, B)$ [a] a group homomorphism
$$(h_*)_p : H_p(X, A) \to H_p(Y, B)$$

3. A function that, for each integer p, assigns to each pair (X, A) in $\mathcal{A}$, a homomorphism
$$(\partial_*)_p : H_p(X, A) \to H_{p-1}(A) \ ,$$
where A denotes the pair $(A, \varnothing)$.

These functions are to satisfy the following axioms, known as the **Eilenberg-Steenrod axioms**, where all pairs of spaces are in $\mathcal{A}$.

[a] h is a continuous function from X to Y such that $h(A) \subseteq B$. By the continuity of $h : (X, A) \to (Y, B)$ we mean the continuity of $h : X \to Y$.

Axiom 1. If i is the identity, then $(i_*)_p$ is the identity for all p.

Axiom 2. Given continuous maps $h : (X, A) \to (Y, B)$ and $k : (Y, B) \to (Z, C)$, $k \circ h : (X, A) \to (Z, C)$ is also continuous. Then
$$((k \circ h)_*)_p = (k_*)_p \circ (h_*)_p$$
for every p.

Axiom 3. If $f : (X, A) \to (Y, B)$ is continuous, then the following diagram commutes:

$$\begin{array}{ccc} H_p(X, A) & \xrightarrow{(f_*)_p} & H_p(Y, B) \\ \Big\downarrow \left(\partial_*^{(X,A)}\right)_p & & \Big\downarrow \left(\partial_*^{(Y,B)}\right)_p \\ H_{p-1}(A) & \xrightarrow[\left(\left(f|_A\right)_*\right)_{p-1}]{} & H_{p-1}(B) \end{array}$$

In other words, $\left(\partial_*^{(Y,B)}\right)_p \circ (f_*)_p = \left(\left(f|_A\right)_*\right)_{p-1} \circ \left(\partial_*^{(X,A)}\right)_p$ for every p.

Axiom 4 (Exactness Axiom)**.** The sequence

$$\cdots \longrightarrow H_{p+1}(X,A) \xrightarrow{(\partial_*)_{p+1}} H_p(A) \xrightarrow{(i_*)_p} H_p(X) \xrightarrow{(\pi_*)_p} H_p(X,A) \xrightarrow{(\partial_*)_p} H_{p-1}(A) \xrightarrow{(i_*)_{p-1}} H_{p-1}(X) \longrightarrow \cdots$$

is exact, where $i : (A, \varnothing) \hookrightarrow (X, \varnothing)$ and $\pi : (X, \varnothing) \hookrightarrow (X, A)$ are inclusions.

(The relation between pairs (X, A) and (X', A') in $\mathcal{A}$ given by $(X', A') \subseteq (X, A)$ means $X' \subseteq X$ and $A' \subseteq A$. The map $i : (X', A') \hookrightarrow (X, A)$ defined by $i(x) = x$ in the case when $(X', A') \subseteq (X, A)$ is called the **inclusion map**.)

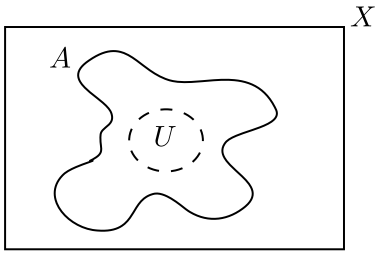
Definition 6.1.3 (Homotopic Maps)**.** Two maps $h, k : (X, A) \to (Y, B)$ are said to be **homotopic** (written $h \simeq p$) is there is a map $F : (X \times I, A \times I) \to (Y, B)$ such that

$$F(x, 0) = h(x) \quad \text{and} \quad F(x, 1) = k(x)$$

for every $x \in X$.

Axiom 5 (Homotopy Axiom)**.** If h and k are homotopic, then $(k_*)_p = (h_*)_p$ for every p.

Axiom 6 (Excision Axiom)**.** Given (X, A) in the class $\mathcal{A}$, let $U \subseteq X$ be an open subset of X such that $\overline{U} \subseteq \operatorname{Int} A$. If $(X \setminus U, A \setminus U)$ is in $\mathcal{A}$, then the inclusion $i : (X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism $(i_*)_p : H_p(X \setminus U, A \setminus U) \xrightarrow{\cong} H_p(X, A)$.



Axiom 7 (Dimension Axiom)**.** If P is a one-point space, then $H_p(P) = 0$ for $p \neq 0$, and $H_0(P) \cong \mathbb{Z}$.

(In general, it can be any abelian group G, called the **coefficient group**. For more details, see page 17 of *Foundations of Algebraic Topology* by Eilenberg and Steenrod.)

Axiom 8 (Axiom of Compact Support)**.** If $\alpha_p \in H_p(X, A)$, there is an admissible pair (X_0, A_0) with X_0 and A_0 compact, such that α_p belongs to the image of the homomorphism $H_p(X_0, A_0) \to H_p(X, A)$ induced by the inclusion $i : (X_0, A_0) \to (X, A)$.

§6.2 The Axioms for Simplicial Theory

We haven't defined homology groups for topological spaces, in general. We did so only for simplicial complexes. Given a polyhedron X, there are many different simplcial complexes whose polytopes equal X. The homology groups of these simplcial complexes are isomorphic to one another in a natural way. But they are, nevertheless, distinct groups. Similarly, if $h : X \to Y$ is a continuous map, where $X = |K|$ and $Y = |L|$ for simplicial complexes K and L, we have defined an induced group homormorphsim $(h_*)_p : H_p(K) \to H_p(L)$.

Of course, if we also have $X = |M|$ and $Y = |N|$ for simplicial complexes M and N that are distinct from K and L, respectively, we also have an induced homomorphism $H_p(M) \to H_p(N)$ that we also denote by $(h_*)_p$. We have come across this notational ambiguity earlier.

The way out of this difficulty is as follows: Given a polyhedron X, we can consider the class of all simplicial complexes that have X as their polytopes, and we can identify the homology groups of these simplicial complexes in a natural way. The resulting homology groups can be called the homology groups of the polyhedron X.

More generally, we can perform the same construction for any topological space that is homeomorphic to a polyhedron.

> **Definition 6.2.1** (Triangulation)**.** Let A be a subspace of a topological space X. A **triangulation** of the pair (X, A) is a simplicial complex K, a subcomplex K_0 of K, and a homeomorphism
>
> $$h : (|K|, |K_0|) \to (X, A) \ .$$
>
> If such a triangulation exists, we say that (X, A) is a **triangulable pair**. If A is empty, we simply say that X is a **triangulable space**.

Now let (X, A) be a triangulable pair. We define the simplicial homology group $H_p(X, A)$ of the pair (X, A) in the following way:

Consider the collection of all triangulations of (X, A). They are of the form

$$h_\alpha : (|K_\alpha|, |C_\alpha|) \to (X, A) \ ,$$

where C_α is a subcomplex of K_α. We can't call the collection of all triangulations of a pair a set in the same spirit as the collection of all sets fails to be a set.

We avoid such problems by assuming that each K_α lies in some fixed generalized Euclidean space $\mathbb{E}^J$. We can do this by taking J to have the cardinality of X itself.

For a fixed p, consider the groups $H_p(K_\alpha, C_\alpha)$. We form $H_p(K_\alpha, C_\alpha) \times \{\alpha\}$. It's evident that $H_p(K_\alpha, C_\alpha) \times \{\alpha\}$ and $H_p(K_\beta, C_\beta) \times \{\beta\}$ are disjoint whenever α and β are distinct. Now we introduce an equivalence relation in the disjoint union

$$\bigsqcup_\alpha H_p(K_\alpha, C_\alpha) \times \{\alpha\} \ .$$

Let $(x_p, \alpha) \in H_p(K_\alpha, C_\alpha) \times \{\alpha\}$ and $(y_p, \beta) \in H_p(K_\beta, C_\beta) \times \{\beta\}$. We define the relation $\sim$ as follows:

$$(x_p, \alpha) \sim (y_p, \beta) \iff \left(\left(h_\beta^{-1} \circ h_\alpha\right)_*\right)_p (x_p) = y_p \, .$$

And we let $H_p(X, A)$ denote the set of equivalence classes. Now, each equivalence class contains exactly one element from each group $H_p(K_\alpha, C_\alpha) \times \{\alpha\}$. That is, the map

$$H_p(K_\alpha, C_\alpha) \times \{\alpha\} \to H_p(X, A)$$

that carries each element to its equivalence class in $H_p(X, A)$ is bijective. We make $H_p(X, A)$ a group by requiring this map to be an isomorphism, *i.e.*, bijective group homomorphism. The group operation in $H_p(X, A)$ reads

$$[x_p] + [y_p] = [x_p + y_p] \ .$$

One can easily check that this is, indeed, a well-defined operation; and this operation makes $H_p(X, A)$ a group.

Now we need to define homomorphism induced by a continuous map. A continuous map $h : (X, A) \to (Y, B)$ induces a homomorphism in homology as follows: take any pair of triangulations

$$h_\alpha : (|K_\alpha|, |C_\alpha|) \to (X, A) \quad \text{and} \quad k_\beta : (|L_\beta|, |D_\beta|) \to (Y, B)$$

Here h_α and k_β are homeomorphisms. The map h induces a map

$$h' : (|K_\alpha|, |C_\alpha|) \to (|L_\beta|, |D_\beta|)$$

Here $h' = k_\beta^{-1} \circ h \circ h_\alpha$, which induces a homomorphism at the level of homology groups:

$$\left(h'_*\right)_p : H_p(K_\alpha, C_\alpha) \to H_p(L_\beta, D_\beta)$$

This yields a well-defined group homomrophism by passing to equivalence classes:

$$\begin{aligned}(h_*)_p : H_p(X, A) &\to H_p(Y, B)\\ [(x_p, \alpha)] &\mapsto \left[\left(\left(h'_*\right)_p x_p, \beta\right)\right]\end{aligned}$$

Now we need to show how we get the boundary homomorphism $\left(\partial_*^X\right)_p : H_p(X, A) \to H_{p-1}(A)$ induced from the simplcial homology boundary homomorphism $\left(\partial_*^{K_\alpha}\right)_p : H_p(K_\alpha, C_\alpha) \to H_{p-1}(C_\alpha)$. $\left(\partial_*^X\right)_p$ is obtained as follows:

$$\left(\partial_*^X\right)_p = \left(\left(h_\alpha\big|_{C_\alpha}\right)_*\right)_{p-1} \circ \left(\partial_*^{K_\alpha}\right)_p \circ \left(\left(h_\alpha^{-1}\right)_*\right)_p ,$$

where $h_\alpha\big|_{C_\alpha}$ is the restriction of $h_\alpha : (|K_\alpha|, |C_\alpha|) \to (X, A)$ to the subcomplex C_α, and $\left(\left(h_\alpha\big|_{C_\alpha}\right)_*\right)_{p-1}$ is the induced homomrophism in dimension $p-1$:

$$\left(\left(h_\alpha\big|_{C_\alpha}\right)_*\right)_{p-1} : H_{p-1}(C_\alpha) \to H_{p-1}(A)$$

Also, h_α is a homeomorphism, so its inverse exists and the inverse $h_\alpha^{-1} : (X, A) \to (|K_\alpha|, |C_\alpha|)$ is continuous, which induces the following group homomorphism:

$$\left(\left(h_\alpha^{-1}\right)_*\right)_p : H_p(X, A) \to H_p(K_\alpha, C_\alpha)$$

Therefore, one can indeed compose the three maps $\left(\left(h_\alpha^{-1}\right)_*\right)_p$, $\left(\partial_*^{K_\alpha}\right)_p$ and $\left(\left(h_\alpha\big|_{C_\alpha}\right)_*\right)_{p-1}$ to obtain

$$\left(\partial_*^X\right)_p = \left(\left(h_\alpha\big|_{C_\alpha}\right)_*\right)_{p-1} \circ \left(\partial_*^{K_\alpha}\right)_p \circ \left(\left(h_\alpha^{-1}\right)_*\right)_p : H_p(X, A) \to H_{p-1}(A)$$

We thus have all the components for a homology theory. First we need to convince ourselves that the class of triangulable pairs forms an admissible class of spaces for a homology theory. It can easily be verified that if (X, A) is triangulable, then so are (X, X), $(X, \varnothing)$, (A, A) and $(A, \varnothing)$. For instance, (X, A) is triangulable implies the existence of a homeomorphism

$$h_\alpha : (|K_\alpha|, |C_\alpha|) \to (X, A) ,$$

with K_α being a simplicial complex and C_α being a subcomplex of K_α. We know from the definition of a map between pair of spaces that $h_\alpha : |K_\alpha| \to X$ is a homeomorphism with $h(|C_\alpha|) \subseteq A$. One, therefore, has the following homeomorphisms:

$$l_\alpha : (|K_\alpha|, |K_\alpha|) \to (X, X) \ , \ m_\alpha : (|K_\alpha|, \varnothing) \to (X, \varnothing) ,$$

$$n_\alpha : (|C_\alpha|, |C_\alpha|) \to (A, A) \ , \ p_\alpha : (|C_\alpha|, \varnothing) \to (A, \varnothing) ,$$

which establishes the fact that (X, X), $(X, \varnothing)$, (A, A) and $(A, \varnothing)$ are triangulable pairs. Any one point space is a 0-simplex, and hence trivially triangulable.

Lemma 6.2.1

If K is a complex, then $|K| \times I$ is the polytope of a complex M, such that each set $\sigma \times I$ is the polytope of a subcomplex of M, and $\sigma \times 0$ and $\sigma \times 1$ are simplices of M, for each simplex σ of K.

Interested readers are encouraged to go through §19 of *Elements of Algebraic Topology* by James Munkres for a proof of Lemma 6.2.1.

By means of this lemma, one finds that if (X, A) is triangulable, then so is $(X \times I, A \times I)$. Given the homeomorphism $h_\alpha : (|K_\alpha|, |C_\alpha|) \to (X, A)$, one can find a homeomorphism

$$t_\alpha : (|K_\alpha| \times I, |C_\alpha| \times I) \to (X \times I, A \times I) \ .$$

All you need to see is that if $h_\alpha : |K_\alpha| \to X$ is a homeomorphism with $h_\alpha(C_\alpha) \subseteq A$, then $t_\alpha : |K_\alpha| \times I \to X \times I$ is a homeomorphism with $t_\alpha(|C_\alpha| \times I) \subseteq A \times I$.

Besides, givens a subcomplex C_α of K_α, $|C_\alpha| \times I$ is the polytope of a subcomplex of a complex whose polytope is $|K_\alpha| \times I$, as guaranteed by Lemma 6.2.1.

Theorem 6.2.2

Simplicial homology theory on the class of triangulable pairs satisfies the Eilenberg-Steenrod axioms.

Proof. ***Axiom 1.*** If $i : (X, A) \to (X, A)$ is the identity, then $(i_*)_p$ is the identity for every p.
Consider a triangulation of (X, A),

$$h_\alpha : (|K_\alpha|, |C_\alpha|) \xrightarrow{\cong} (X, A)$$

Then $h_\alpha^{-1} \circ i \circ h_\alpha = i' : (|K_\alpha|, |C_\alpha|) \to (|K_\alpha|, |C_\alpha|)$. i is an identity map. So we have

$$i' = h_\alpha^{-1} \circ i \circ h_\alpha = h_\alpha^{-1} \circ h_\alpha = \mathrm{id}_{(|K_\alpha|, |C_\alpha|)}$$

So i' is the identity map of $(|K_\alpha|, |C_\alpha|)$. By *Theorem 18.3* of *Elements of Algebraic Topology*, $(i'_*)_p : H_p(|K_\alpha|, |C_\alpha|) \to H_p(|K_\alpha|, |C_\alpha|)$ is the identity.

A generic element of $H_p(X, A)$ is the equivalence class of (x_p, α), where $x_p \in H_p(K_\alpha, C_\alpha)$. $(i'_*)_p$ is the identity, so $(i'_*)_p x_p = x_p$

$$(i_*)_p [(x_p, \alpha)] = \left[\left((i'_*)_p x_p, \alpha\right)\right] = [(x_p, \alpha)]$$

So $(i_*)_p : H_p(X, A) \to H_p(X, A)$ is the identity.

Axiom 2. $((k \circ h)_*)_p = (k_*)_p \circ (h_*)_p$.
Suppose $h : (X, A) \to (Y, B)$ and $k : (Y, B) \to (Z, C)$ are continuous, then the composition $k \circ h : (X, A) \to (Z, C)$ is also continuous. Consider triangulations of the spaces $(X, A), (Y, B), (Z, C)$.

$$t_\alpha : (|K_\alpha|, |C_\alpha|) \xrightarrow{\cong} (X, A) \ , \ t_\beta : (|L_\beta|, |D_\beta|) \xrightarrow{\cong} (Y, B) \ , \ t_\gamma : (|M_\gamma|, |E_\gamma|) \xrightarrow{\cong} (Z, C)$$

Let $k \circ h = l$. Then

$$h' = t_\beta^{-1} \circ h \circ t_\alpha \ , \ \ k' = t_\gamma^{-1} \circ k \circ t_\beta \ , \ \ l' = t_\gamma^{-1} \circ l \circ t_\alpha = k' \circ h'$$

k', h', l' are continuous maps between pairs of simplicial complexes. So by *Theorem 18.3* of *Elements of Algebraic Topology*, $(l'_*)_p = (k'_*)_p \circ (h'_*)_p$.

Let's take $[(x_p, \alpha)] \in H_p(X, A)$, with $x_p \in H_p(K_\alpha, C_\alpha)$.

$$\begin{aligned} (l_*)_p [(x_p, \alpha)] &= \left[\left((l'_*)_p x_p, \gamma\right)\right] = \left[\left(\left((k'_*)_p \circ (h'_*)_p\right) x_p, \gamma\right)\right] \\ &= (k_*)_p \left[\left((h'_*)_p x_p, \beta\right)\right] = \left((k_*)_p \circ (h_*)_p\right) [(x_p, \alpha)] \end{aligned}$$

Therefore, $(l_*)_p = (k_*)_p \circ (h_*)_p$. And $l = k \circ h$, hence $((k \circ h)_*)_p = (k_*)_p \circ (h_*)_p$.

Axiom 3. If $f : (X, A) \to (Y, B)$ is continuous, then the following diagram commutes:

$$\begin{array}{ccc} H_p(X,A) & \xrightarrow{(f_*)_p} & H_p(Y,B) \\ \Big\downarrow \left(\partial_*^{(X,A)}\right)_p & & \Big\downarrow \left(\partial_*^{(Y,B)}\right)_p \\ H_{p-1}(A) & \xrightarrow[\left(\left(f\big|_A\right)_*\right)_{p-1}]{} & H_{p-1}(B) \end{array}$$

In other words, $\left(\partial_*^{(Y,B)}\right)_p \circ (f_*)_p = \left(\left(f\big|_A\right)_*\right)_{p-1} \circ \left(\partial_*^{(X,A)}\right)_p$.

Consider triangulations of (X,A) and (Y,B):

$$h_\alpha : (|K_\alpha|, |C_\alpha|) \xrightarrow{\cong} (X,A) \ , \ k_\beta : (|L_\beta|, |D_\beta|) \xrightarrow{\cong} (Y,B)$$

Then $k_\beta^{-1} \circ f \circ h_\alpha = f' : (|K_\alpha|, |C_\alpha|) \to (|L_\beta|, |D_\beta|)$. f' is a continuous map from $|K_\alpha|$ to $|L_\beta|$. So it has a simplicial approximation $f'' : K'_\alpha \to L_\beta$, where K'_α is a subdivision of K_α.

$f' : |K'_\alpha| \to |L_\beta|$ maps $|C_\alpha|$ into $|D_\beta|$. So its simplicial approximation f'' also maps $|C_\alpha|$ into $|D_\beta|$ (by *Lemma 14.4(a)*). In other words, C_α has a subdivision C'_α that gets mapped to D_β by f''. Therefore, $f'' : (K'_\alpha, C'_\alpha) \to (L_\beta, D_\beta)$ is a simplicial map.

Let $f'\big|_{C_\alpha} = g'$ and $f''\big|_{C_{\alpha'}} = g''$. By the definition of homomorphism induced by a continuous map between simplicial complexes, $(g'_*)_p = (g''_*)_p \circ (\lambda'_*)_p$, where $\lambda' : \mathcal{C}(C_\alpha) \to \mathcal{C}(C'_\alpha)$ is the restriction of subdivision operator $\lambda : \mathcal{C}(K_\alpha) \to \mathcal{C}(K'_\alpha)$ (existence of subdivision operator is ensured by *Theorem 17.2*).

$$(f'_*)_p = (f''_*)_p \circ \left(\widetilde{\lambda}_*\right)_p$$

where $\widetilde{\lambda}_p : C_p(K_\alpha, C_\alpha) \to C_p(K'_\alpha, C'_\alpha)$ is the chain map induced by the subdivision operator (existence of λ_p is guaranteed by *Theorem 17.3*).

First we want to show that the following diagram commutes:

$$\begin{array}{ccc} H_p(K_\alpha, C_\alpha) & \xrightarrow{(f'_*)_p} & H_p(L_\beta, D_\beta) \\ \Big\downarrow (\partial_*^K)_p & & \Big\downarrow (\partial_*^L)_p \\ H_{p-1}(C_\alpha) & \xrightarrow[\left(\left(f'\big|_{C_\alpha}\right)_*\right)_{p-1}]{} & H_{p-1}(D_\beta) \end{array}$$

Let's take $\{d_p + C_p(C_\alpha)\} \in H_p(K_\alpha, C_\alpha)$.

$$\{d_p + C_p(C_\alpha)\} \xmapsto{(\widetilde{\lambda}_*)_p} \left\{\widetilde{\lambda}_p(d_p + C_p(C_\alpha))\right\} = \left\{\lambda_p(d_p) + C_p\left(C'_\alpha\right)\right\}$$

$$\left\{\lambda_p(d_p) + C_p\left(C'_\alpha\right)\right\} \xmapsto{(f''_*)_p} \left\{\left(f''_\#\right)_p(\lambda_p(d_p)) + C_p(L_\beta)\right\}$$

$$\left\{\left(f''_\#\right)_p(\lambda_p(d_p)) + C_p(L_\beta)\right\} \xmapsto{(\partial_*^L)_p} \left\{\partial_p^L \left(f''_\#\right)_p(\lambda_p(d_p))\right\}$$

$$\therefore \left((\partial_*^L)_p \circ (f'_*)_p\right)\{d_p + C_p(C_\alpha)\} = \left\{\partial_p^L \left(f''_\#\right)_p(\lambda_p(d_p))\right\}$$

f'' is a simplicial map, so $f''_\#$ is a chain map. Therefore, $\partial_p^L \circ \left(f''_\#\right)_p = \left(f''_\#\right)_{p-1} \circ \partial_p^{K'}$, where $\partial_p^{K'}$ is the boundary operator in K'_α. Furthermore, λ is also a chain map, so $\partial_p^{K'} \circ \lambda_p = \lambda_{p-1} \circ \partial_p^K$. Therefore,

$$\left\{\partial_p^L \left(f''_\#\right)_p(\lambda_p(d_p))\right\} = \left\{\left(f''_\#\right)_{p-1}\left(\lambda_{p-1}\left(\partial_p^K d_p\right)\right)\right\}$$

$\partial_p^K d_p$ is carried by C_α as proved in lecture while defining homology boundary homomorphism. So $\partial_p^K d_p \in C_{p-1}(C_\alpha)$. λ'_{p-1} is the restriction of λ_{p-1} on $C_{p-1}(C_\alpha)$, so

$$\lambda_{p-1}\left(\partial_p^K d_p\right) = \lambda'_{p-1}\left(\partial_p^K d_p\right) \in C_{p-1}\left(C'_\alpha\right)$$

g'' is the restriction of f'' on C'_α. So $g''_\#$ is the restriction of $f''_\#$ on $\mathcal{C}\left(C'_\alpha\right)$. Hence,

$$\left(f''_\#\right)_{p-1}\left(\lambda'_{p-1}\left(\partial_p^K d_p\right)\right) = \left(g''_\#\right)_{p-1}\left(\lambda'_{p-1}\left(\partial_p^K d_p\right)\right)$$

$$\therefore \left\{\left(f''_\#\right)_{p-1}\left(\lambda_{p-1}\left(\partial_p^K d_p\right)\right)\right\} = \left\{\left(g''_\#\right)_{p-1}\left(\lambda'_{p-1}\left(\partial_p^K d_p\right)\right)\right\} = \left(g''_*\right)_{p-1}\left\{\lambda'_{p-1}\left(\partial_p^K d_p\right)\right\}$$

$$\left\{\lambda'_{p-1}\left(\partial_p^K d_p\right)\right\} = \left(\lambda'_*\right)_{p-1}\left\{\partial_p^K d_p\right\}$$

$$\implies \left\{\left(f''_\#\right)_{p-1}\left(\lambda_{p-1}\left(\partial_p^K d_p\right)\right)\right\} = \left(\left(g''_*\right)_{p-1} \circ \left(\lambda'_*\right)_{p-1}\right)\left\{\partial_p^K d_p\right\} = \left(g'_*\right)_{p-1}\left\{\partial_p^K d_p\right\}$$

And $\left(\partial_*^K\right)_p\left\{d_p + C_p\left(C_\alpha\right)\right\} = \left\{\partial_p^K d_p\right\}$. Therefore,

$$\left(\partial_*^L\right)_p \circ \left(f'_*\right)_p = \left(g'_*\right)_{p-1} \circ \left(\partial_*^K\right)_p = \left(\left(f'\big|_{C_\alpha}\right)_*\right)_{p-1} \circ \left(\partial_*^K\right)_p$$

Now we proceed on to proving the general statement. $f = k_\beta \circ f' \circ h_\alpha^{-1}$, and $f\big|_A = k_\beta\big|_{D_\beta} \circ f'\big|_{C_\alpha} \circ h_\alpha\big|_{C_\alpha}^{-1}$.

$$\left(\partial_*^{(Y,B)}\right)_p \circ \left(f_*\right)_p = \left(\left(k_\beta\big|_{D_\beta}\right)_*\right)_{p-1} \circ \left(\partial_*^L\right)_p \circ \left(\left(k_\beta^{-1}\right)_*\right)_p \circ \left(\left(k_\beta\right)_*\right)_p \circ \left(f'_*\right)_p \circ \left(\left(h_\alpha^{-1}\right)_*\right)_p$$

$$\left(\left(k_\beta^{-1}\right)_*\right)_p \circ \left(\left(k_\beta\right)_*\right)_p = \left(\left(k_\beta^{-1} \circ k_\beta\right)_*\right)_p = \left(\left(\mathrm{id}_{(|L_\beta|,|D_\beta|)}\right)_*\right)_p = \mathrm{id}_{H_p(L_\beta, D_\beta)}$$

$$\therefore \left(\partial_*^{(Y,B)}\right)_p \circ \left(f_*\right)_p = \left(\left(k_\beta\big|_{D_\beta}\right)_*\right)_{p-1} \circ \left(\partial_*^L\right)_p \circ \left(f'_*\right)_p \circ \left(\left(h_\alpha^{-1}\right)_*\right)_p$$

$$\left(\left(f\big|_A\right)_*\right)_{p-1} \circ \left(\partial_*^{(X,A)}\right)_p = \left(\left(k_\beta\big|_{D_\beta}\right)_*\right)_{p-1} \circ \left(\left(f'\big|_{C_\alpha}\right)_*\right)_{p-1} \circ \left(\left(h_\alpha\big|_{C_\alpha}^{-1}\right)_*\right)_{p-1} \circ \left(\left(h_\alpha\big|_{C_\alpha}\right)_*\right)_{p-1} \circ \left(\partial_*^K\right)_p \circ \left(\left(h_\alpha^{-1}\right)_*\right)_p$$

Similar as above, $\left(\left(h_\alpha\big|_{C_\alpha}^{-1}\right)_*\right)_{p-1} \circ \left(\left(h_\alpha\big|_{C_\alpha}\right)_*\right)_{p-1} = \mathrm{id}_{H_{p-1}(C_\alpha)}$. Therefore,

$$\left(\left(f\big|_A\right)_*\right)_{p-1} \circ \left(\partial_*^{(X,A)}\right)_p = \left(\left(k_\beta\big|_{D_\beta}\right)_*\right)_{p-1} \circ \left(\left(f'\big|_{C_\alpha}\right)_*\right)_{p-1} \circ \left(\partial_*^K\right)_p \circ \left(\left(h_\alpha^{-1}\right)_*\right)_p$$

We proved that $\left(\partial_*^L\right)_p \circ \left(f'_*\right)_p = \left(\left(f'\big|_{C_\alpha}\right)_*\right)_{p-1} \circ \left(\partial_*^K\right)_p$. Combining this with the expressions of $\left(\partial_*^{(Y,B)}\right)_p \circ \left(f_*\right)_p$ and $\left(\left(f\big|_A\right)_*\right)_{p-1} \circ \left(\partial_*^{(X,A)}\right)_p$, we get that

$$\left(\partial_*^{(Y,B)}\right)_p \circ \left(f_*\right)_p = \left(\left(f\big|_A\right)_*\right)_{p-1} \circ \left(\partial_*^{(X,A)}\right)_p$$

So the diagram commutes.

Axiom 4. The sequence

$$\cdots \longrightarrow H_{p+1}(X,A) \xrightarrow{(\partial_*)_{p+1}} H_p(A) \xrightarrow{(i_*)_p} H_p(X) \xrightarrow{(\pi_*)_p} H_p(X,A) \xrightarrow{(\partial_*)_p} H_{p-1}(A) \xrightarrow{(i_*)_{p-1}} H_{p-1}(X) \longrightarrow \cdots$$

is exact, where $i : (A, \varnothing) \to (X, \varnothing)$ and $\pi : (X, \varnothing) \to (X, A)$ are inclusions.
Consider a triangulation of (X, A):

$$h_\alpha : (|K_\alpha|, |C_\alpha|) \xrightarrow{\cong} (X, A)$$

The restriction of h_α gives rise to triangulation of X and A:

$$h'_\alpha : (|K_\alpha|, \varnothing) \xrightarrow{\cong} (X, \varnothing) \quad \text{and } h''_\alpha : (|C_\alpha|, \varnothing) \xrightarrow{\cong} (A, \varnothing)$$

By *Zig-Zag lemma*, the following sequence is exact:

$$\cdots \longrightarrow H_{p+1}(K_\alpha, C_\alpha) \xrightarrow{(\partial^K_*)_{p+1}} H_p(C_\alpha) \xrightarrow{(i'_*)_p} H_p(K_\alpha) \xrightarrow{(\pi'_*)_p} H_p(K_\alpha, C_\alpha) \xrightarrow{(\partial^K_*)_p} H_{p-1}(C_\alpha) \longrightarrow \cdots$$

where $i' = (h'_\alpha)^{-1} \circ i \circ h''_\alpha : (|C_\alpha|, \varnothing) \to (|K_\alpha|, \varnothing)$ and $\pi' = h_\alpha^{-1} \circ \pi \circ h'_\alpha : (|K_\alpha|, \varnothing) \to (|K_\alpha|, |C_\alpha|)$ are inclusions.

Claim — In the following commutative diagram,

$$\begin{array}{ccccc} A & \xrightarrow{f_1} & B & \xrightarrow{f_2} & C \\ {\scriptstyle i_1}\uparrow & & {\scriptstyle i_2}\uparrow & & {\scriptstyle i_3}\uparrow \\ D & \xrightarrow{g_1} & E & \xrightarrow{g_2} & F \end{array}$$

if the lower sequence is exact at E, and the vertical maps are isomorphisms, then the upper sequence is exact at B.

Proof. $\operatorname{im} g_1 = \operatorname{Ker} g_2$ gives us $g_2 \circ g_1 = 0$. Then

$$f_2 \circ f_1 = i_3 \circ g_2 \circ g_1 \circ i_1^{-1} = 0$$

So $\operatorname{im} f_1 \subseteq \operatorname{Ker} f_2$. Now let $b \in \operatorname{Ker} f_2$. Then $f_2(b) = 0$. So

$$g_2\left(i_2^{-1}(b)\right) = i_3^{-1}(f_2(b)) = i_3^{-1}(0) = 0$$

So $i_2^{-1}(b) \in \operatorname{Ker} g_2 = \operatorname{im} g_1$. In other words, $i_2^{-1}(b) = g_1(d)$ for some $d \in D$.

$$f_1(i_1(d)) = i_2(g_1(d)) = i_2\left(i_2^{-1}(b)\right) = b$$

So $b \in \operatorname{im} f_1$. Therefore, $\operatorname{im} f_1 = \operatorname{Ker} f_2$. □

Now consider the following diagram:

$$\begin{array}{ccccccccc} H_{p+1}(X, A) & \xrightarrow{(\partial_*)_{p+1}} & H_p(A) & \xrightarrow{(i_*)_p} & H_p(X) & \xrightarrow{(\pi_*)_p} & H_p(X, A) & \xrightarrow{(\partial_*)_p} & H_{p-1}(A) \\ {\scriptstyle ((h_\alpha)_*)_{p+1}}\uparrow & & {\scriptstyle ((h''_\alpha)_*)_p}\uparrow & & {\scriptstyle ((h'_\alpha)_*)_p}\uparrow & & {\scriptstyle ((h_\alpha)_*)_p}\uparrow & & {\scriptstyle ((h''_\alpha)_*)_{p-1}}\uparrow \\ H_{p+1}(K_\alpha, C_\alpha) & \xrightarrow[(\partial^K_*)_{p+1}]{} & H_p(C_\alpha) & \xrightarrow[(i'_*)_p]{} & H_p(K_\alpha) & \xrightarrow[(\pi'_*)_p]{} & H_p(K_\alpha, C_\alpha) & \xrightarrow[(\partial^K_*)_p]{} & H_{p-1}(C_\alpha) \end{array}$$

In this diagram, the lower sequence is exact, we need to show that the upper sequence is exact. This diagram is easily seen to be commutative, using Axiom 2 and definition of $(\partial_*)_p$. Furthermore, $((h_\alpha)_*)_p$ is an isomorphism. Because

$$((h_\alpha)_*)_p \circ ((h_\alpha^{-1})_*)_p = ((h_\alpha \circ h_\alpha^{-1})_*)_p = \left(\left(\mathrm{id}_{(X,A)}\right)_*\right)_p = \mathrm{id}_{H_p(X,A)}$$

And similarly,

$$((h_\alpha^{-1})_*)_p \circ ((h_\alpha)_*)_p = ((h_\alpha^{-1} \circ h_\alpha)_*)_p = \left(\left(\mathrm{id}_{(|K_\alpha|,|C_\alpha|)}\right)_*\right)_p = \mathrm{id}_{H_p(K_\alpha,C_\alpha)}$$

Therefore, $((h_\alpha)_*)_p$ is an isomorphism. Similarly, $((h'_\alpha)_*)_p$ and $((h''_\alpha)_*)_p$ are also isomorphisms. Therefore, using the claim, we can conclude that the upper sequence is exact.

Axiom 5. If $h, k : (X, A) \to (Y, B)$ are homotopic, then $(h_*)_p = (k_*)_p$ for every p.
Consider trinagulations of (X, A) and (Y, B).

$$t_\alpha : (|K_\alpha|, |C_\alpha|) \xrightarrow{\cong} (X, A) \ , \ t_\beta : (|L_\beta|, |D_\beta|) \xrightarrow{\cong} (Y, B)$$

Then $h' = t_\beta^{-1} \circ h \circ t_\alpha$, $k' = t_\beta^{-1} \circ k \circ t_\alpha$ are continuous maps from $(|K_\alpha|, |C_\alpha|)$ to $(|L_\beta|, |D_\beta|)$.

Let F be a homotopy between h and k. In other words, $F : (X \times I, A \times I) \to (Y, B)$ is continuous with the property that

$$F(x, 0) = h(x) \quad \text{and} \quad F(x, 1) = k(x)$$

We claim that $h' \simeq k'$. Consider the map $F' : (|K_\alpha| \times I, |C_\alpha| \times I) \to (|L_\beta|, |D_\beta|)$ defined by

$$F'(x, t) = t_\beta^{-1}(F(t_\alpha(x), t))$$

Then we have

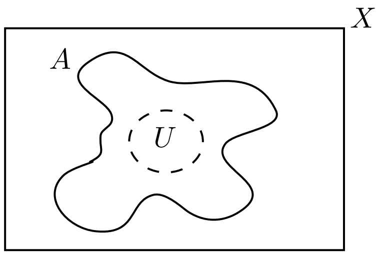
$$F'(x, 0) = t_\beta^{-1}(F(t_\alpha(x), 0)) = t_\beta^{-1}(h(t_\alpha(x))) = h'(x)$$
$$F'(x, 1) = t_\beta^{-1}(F(t_\alpha(x), 1)) = t_\beta^{-1}(k(t_\alpha(x))) = k'(x)$$

So $h' \simeq k'$. Therefore, by *Theorem 19.3*, $(h'_*)_p = (k'_*)_p$. By axiom 2,

$$(h_*)_p = (t_\beta)_* \circ (h'_*)_p \circ ((t_\alpha^{-1})_*)_p \quad \text{and} \quad (k_*)_p = (t_\beta)_* \circ (k'_*)_p \circ ((t_\alpha^{-1})_*)_p$$

Therefore, $(h_*)_p = (k_*)_p$.

Axiom 6. Given (X, A) in the class $\mathcal{A}$, let $U \subseteq X$ be an open subset of X such that $\overline{U} \subseteq \operatorname{Int} A$. If $(X \setminus U, A \setminus U)$ is in $\mathcal{A}$, then the inclusion $i : (X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism $(i_*)_p : H_p(X \setminus U, A \setminus U) \to H_p(X, A)$.
The problem lies in the observation that even though both (X, A) and $(X \setminus U, A \setminus U)$ may be triangulable, the triangulations may be entirely unrelated to one another! If they are related, then the excision axiom follows readily from Theorem 3.1.1.

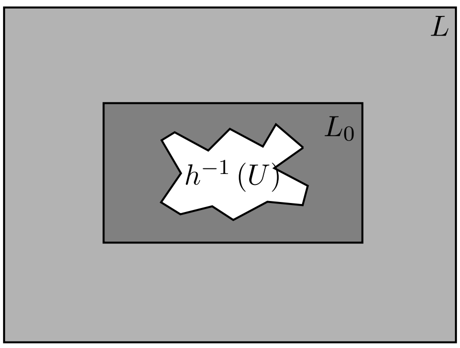


Let $U \subseteq A \subseteq X$. Suppose there is a triangulation

$$h : (|K|, |K_0|) \to (X, A)$$

of the pair (X, A) that induces a triangulation of the subspace $X \setminus U$. This means that $X \setminus U = h(|L|)$ for some subcomplex L of K. Let $L_0 = L \cap K_0$. Then one can easily check that $A \setminus U = h(|L_0|)$.

Now we have the setup to apply Theorem 3.1.1.



We have a triangulation of (X, A)

$$h : (|K|, |K_0|) \to (X, A)$$

that induces a triangulation of $(X \setminus U, A \setminus U)$, which we denote by the same notation:

$$h : (|L|, |L_0|) \to (X \setminus U, A \setminus U)$$

These two maps are homeomorphisms, therefore we have two isomorphisms at the level of homology groups, which we again denote by the same notation:

$$(h_*)_p : H_p(K, K_0) \xrightarrow{\cong} H_p(X, A) \text{ and } (h_*)_p : H_p(L, L_0) \xrightarrow{\cong} H_p(X \setminus U, A \setminus U)$$

The polytope of L is $|K| \setminus h^{-1}(U)$, because $h(|L|) = X \setminus U = h(|K|) \setminus U$. Similarly, the polytope of L_0 is $|K_0| \, h^{-1}(U)$, where $h^{-1}(U)$ is an open set contained in $|K_0|$, because

$$U \subseteq A \implies h^{-1}(U) \subseteq h^{-1}(A) = |K_0|$$

Therefore, by Theorem 3.1.1, the inclusions $L_0 \hookrightarrow L$, $K_0 \hookrightarrow K$ induce the isomorphism

$$H_p(L, L_0) \cong H_p(K, K_0)$$

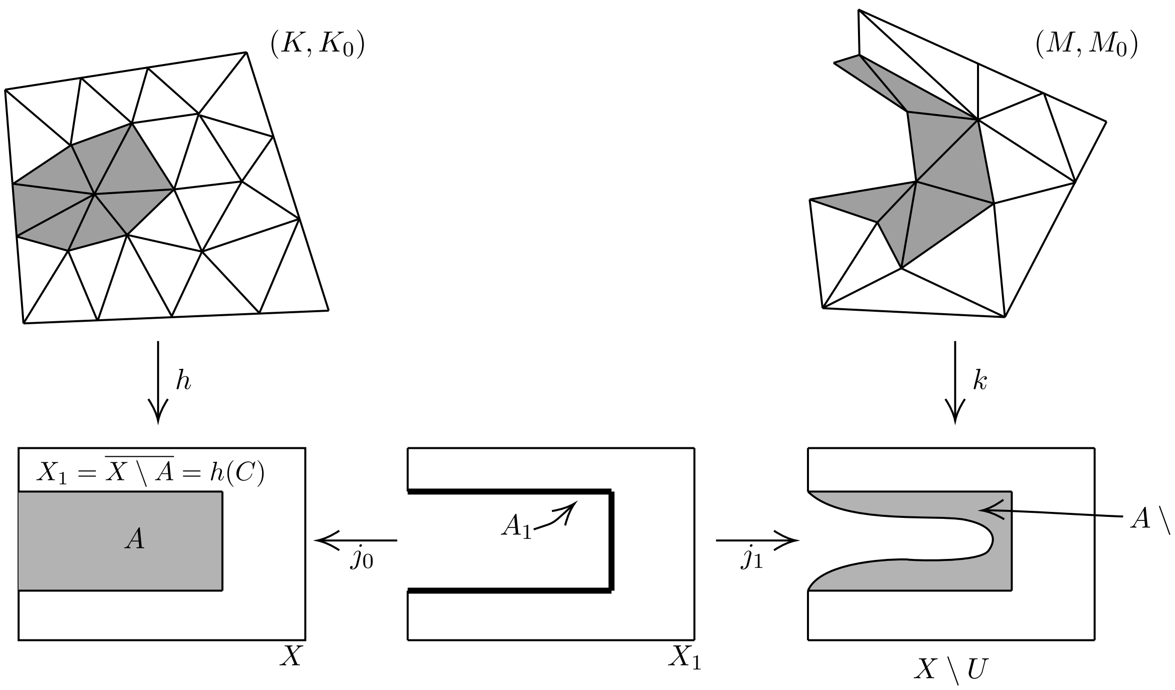
We have just seen that $H_p(K, K_0) \cong H_p(X, A)$ and $H_p(L, L_0) \cong H_p(X \setminus U, A \setminus U)$. Therefore,

$$H_p(X, A) \cong H_p(X \setminus U, A \setminus U) \ .$$

Now we prove this result in a general settiing. Let $U \subseteq \operatorname{Int} A$. Let

$$h : (|K|, |K_0|) \to (X, A) \quad \text{and} \quad k : (|M|, |M_0|) \to (X \setminus U, A \setminus U)$$

be triangulations of the pairs (X, A) and $X \setminus U, A \setminus U$, respectively. Let $X_1 = \overline{X \setminus A}$ and $A_1 = X_1 \cap A$. We claim that the pair (X_1, A_1) is triangulable both by h and k. See the figure below where the maps j_0 and j_1 denote inclusions:



Note that $|K| \setminus |K_0|$ is the union of all open simplices $\operatorname{Int} \sigma$ such that $\sigma \in K$ and $\sigma \notin K_0$. Then its closure $\overline{|K| \setminus |K_0|}$ denoted by C is the polytope of the subcomplex of K consisting of all simplices σ of K that are not in K_0, and their faces.

$$h(C) = h\left(\overline{|K| \setminus |K_0|}\right) = \overline{h(|K|) \setminus h(|K_0|)} = \overline{X \setminus A} = X_1$$

Therefore, X_1 is triangulable by h. Besides, A is also triangulable by h, since $h(|K_0|) = A$. Therefore, $X_1 \cap A$ is also triangulable by the same homeomorphism h. In particular, since h is injective,

$$h(C \cap |K_0|) = h(C) \cap h(|K_0|) = X_1 \cap A = A_1$$

Therefore, $h : (C, C \cap |K_0|) \to (X_1, A_1)$ is a homeomorphism.

Similarly, for the other pair $(|M|, |M_0|)$, the closure of $|M| \setminus |M_0|$ is the polytope of a subcomplex of M. Let us denote $\overline{|M| \setminus |M_0|}$ by D. Then

$$k(D) = k\left(\overline{|M| \setminus |M_0|}\right) = \overline{k(|M|) \setminus k(|M_0|)} = \overline{(X \setminus U) \setminus (A \setminus U)} = \overline{X \setminus A} = X_1$$

So X_1 is triangulable by k. Now notice that, since $U \subseteq \operatorname{Int} A$,

$$A_1 = X_1 \cap (A \setminus U) = k(D) \cap k(|M_0|) = k(D \cap |M_0|)$$

Hence, A_1 is triangulable by k. In other words, $k : (D, D \cap |M_0|) \to (X_1, A_1)$ is a homeomorphism. Therefore, the pair (X_1, A_1) is triangulable by both h and k.

We have two inclusions in this case:

$$j_0 : (X_1, A_1) \hookrightarrow (X, A) \quad \text{and} \quad j_1 : (X_1, A_1) \hookrightarrow (X \setminus U, A \setminus U)$$

h is a triangulation of (X, A), which induces a triangulation of (X_1, A_1). X_1 is the closure of $X \setminus A$, so X_1 is closed in X. Hence, $X_1 = X \setminus V$ for some open set V. Also, $A_1 = X_1 \cap A = (X \setminus V) \cap A = A \setminus V$. Therefore, by the special case we proved earlier, j_0 induces an isomorphism:

$$((j_0)_*)_p : H_p(X_1, A_1) = H_p(X \setminus V, A \setminus V) \xrightarrow{\cong} H_p(X, A) \ .$$

$X_1 = X \setminus V \subseteq X \setminus U$, so $U \subseteq V$. Therefore, $X \setminus V = (X \setminus U) \setminus (V \setminus U)$. X_1 is closed in $X \setminus U$, so $V \setminus U$ is open in $X \setminus U$. Also,

$$A_1 = X_1 \cap (A \setminus U) = ((X \setminus U) \setminus (V \setminus U)) \cap (A \setminus U) = (A \setminus U) \setminus (V \setminus U)$$

Therefore, by the special case we proved earlier, j_1 induces an isomorphism:

$$((j_1)_*)_p : H_p(X_1, A_1) = H_p((X \setminus U) \setminus (V \setminus U), (A \setminus U) \setminus (V \setminus U)) \xrightarrow{\cong} H_p(X \setminus U, A \setminus U) \ .$$

Together they imply

$$H_p(X, A) \cong H_p(X \setminus U, A \setminus U)$$

Axiom 7. If P is a one-point space, then $H_p(P) = 0$ for $p \neq 0$, and $H_0(P) \cong \mathbb{Z}$.

Let $P = \{p\}$. Then P is homeomorphic to the complex K_α containing only one vertex v_0. Consider a triangulation

$$h_\alpha : K_\alpha \xrightarrow{\cong} P$$

At the end of Axiom 4, we proved that homeomorphism induces isomorphism betweem homology groups. Therefore,

$$((h_\alpha)_*)_p : H_p(K_\alpha) \to H_p(P) \ \text{ is an isomorphism.}$$

K_α does not contain any p-dimensional simplices for $p > 0$. So $H_p(K_\alpha) = 0$ for $p \neq 0$. And $H_0(K_\alpha) \cong \mathbb{Z}$ since K_α is connected. Therefore, $H_p(P) = 0$ for $p \neq 0$; and $H_0(P) \cong H_0(K_\alpha) \cong \mathbb{Z}$.

Axiom 8. If $\alpha_p \in H_p(X, A)$, there is an admissible pair (X_0, A_0) with X_0 and A_0 compact, such that α_p belongs to the image of the homomorphism $H_p(X_0, A_0) \to H_p(X, A)$ induced by the inclusion $i : (X_0, A_0) \to (X, A)$.

Consider a triangulation of (X, A):

$$t_\alpha : (|K_\alpha|, |C_\alpha|) \xrightarrow{\cong} (X, A)$$

Take an element $[(z,\alpha)] \in H_p(X,A)$, where $z \in H_p(K_\alpha, C_\alpha)$. Now, $z = \{c_p + C_p(C_\alpha)\}$ for some $c_p \in C_p(K)$; and $\partial_p c_p$ is carried by C_α. We know that c_p can be written uniquely as a finite linear combination of elementary p-chains. Therefore, c_p is carried by a finite subcomplex L_α of K_α. So c_p can be considered as a cycle of (L_α, D_α), where $D_\alpha = L_\alpha \cap C_\alpha$.

L_α is a finite complex, so by *Lemma 2.5*, $|L_\alpha|$ is compact. Similarly, $|D_\alpha|$ is also compact. As a result, $X_0 = t_\alpha(|L_\alpha|)$ and $A_0 = t_\alpha(|D_\alpha|)$ are also compact.

If we let $t_{\widetilde{\alpha}} : (|L_\alpha|, |D_\alpha|) \to (X_0, A_0)$ be the restriction of t_α, then $t_{\widetilde{\alpha}}$ is a triangulation of (X_0, A_0). Let $j : (X_0, A_0) \to (X, A)$ be the inclusion map. Then $j' = t_\alpha^{-1} \circ j \circ t_{\widetilde{\alpha}} : (L_\alpha, D_\alpha) \to (K_\alpha, C_\alpha)$ is the inclusion map.

c_p is carried by L_α, and $\partial_p c_p$ is carried by $L_\alpha \cap C_\alpha = D_\alpha$. So the homology class of c_p in $H_p(L_\alpha, D_\alpha)$ is $\{c_p + C_p(D_\alpha)\}$.

$$\{c_p + C_p(D_\alpha)\} \xmapsto{(j'_*)_p} \{c_p + C_p(C_\alpha)\}$$

Now let $w = \{c_p + C_p(D_\alpha)\} \in H_p(L_\alpha, D_\alpha)$. Since $t_{\widetilde{\alpha}}$ is a triangulation of (X_0, A_0), $[(w, \widetilde{\alpha})] \in H_p(X_0, A_0)$.

$$(j_*)_p [(w, \widetilde{\alpha})] = \left[\left(\left(j'_*\right)_p w, \alpha\right)\right] = [(z, \alpha)]$$

Therefore, $[(z,\alpha)]$ lies in the image of the homomorphism induced by the inclusion $j : (X_0, A_0) \to (X, A)$; and (X_0, A_0) is a compact admissible pair. ■