Differential Geometry II (MAT401)

Lecture Notes

Preface

This series of lecture notes has been prepared for aiding students who took the BRAC University course **Differential Geometry II (MAT401)** in Fall 2023 semester. These notes were typeset under the supervision of mathematician **Dr. Syed Hasibul Hassan Chowdhury**. The main goal of this typeset is to have an organized digital version of the notes, which is easier to share and handle. If you see any mistakes or typos, please send me an email at atonuroychowdhury@gmail.com

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References:

- *An Introduction to Manifolds*, by **Loring W. Tu**
- *Analysis on Manifolds*, by **James R. Munkres**
- *An Introduction to Differentiable Manifolds and Riemannian Geometry*, by **William Boothby**
- *Geometry of Differential Forms*, by **Shigeyuki Morita**
- *From Calculus to Cohomology: De Rham Cohomology and Characteristic Classes*, by **Ib Madsen and Jxrgen Tornehave**
- *Differential Forms*, by **Victor Guillemin and Peter Haine**.

Contents

1 Review of Multilinear Algebra

§1.1 Dual Space

Let V and W be real vector spaces. We denote by $\mathrm{Hom}(V, W)$ the vector space of all linear maps $f : V \to W$. In particular, if we choose $W = \mathbb{R}$, we get the **dual space** V^*.

$$V^* = \mathrm{Hom}(V, \mathbb{R}).$$

The elements of V^* are called covectors on V. In the rest of the lecture, we will assume V to be a finite dimensional vector space. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be a basis for V. Then every $\mathbf{v} \in V$ is a unique linear combination

$$\mathbf{v} = \sum_{i=1}^{n} v^i \mathbf{e}_i, \tag{1.1}$$

with $v^i \in \mathbb{R}$. v^i's are called the coordinates of $\mathbf{v}$ relative to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. Let $\widehat{\alpha}^i$ be the linear function on V that picks up the i-th coordinate of the vector, i.e.

$$\widehat{\alpha}^i(\mathbf{v}) = \widehat{\alpha}^i\left(\sum_{i=1}^{n} v^i \mathbf{e}_i\right) = v^i. \tag{1.2}$$

When $\mathbf{v}$ is one of the basis vectors,

$$\widehat{\alpha}^i(\mathbf{e}_j) = \delta^i{}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1.3}$$

Proposition 1.1

The functions $\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n$ form a basis for V^*.

Proof. Suppose $f \in V^*$. Then for any $\mathbf{v} = \sum_{i=1}^{n} v^i \mathbf{e}_i \in V$,

$$f(\mathbf{v}) = f\left(\sum_{i=1}^{n} v^i \mathbf{e}_i\right) = \sum_{i=1}^{n} v^i f(\mathbf{e}_i) = \sum_{i=1}^{n} f(\mathbf{e}_i)\, \widehat{\alpha}^i(\mathbf{v}).$$

Since this holds for any $\mathbf{v} \in V$,

$$f = \sum_{i=1}^{n} f(\mathbf{e}_i)\, \widehat{\alpha}^i. \tag{1.4}$$

Therefore, $\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n$ span V^*. As for linear independence, suppose

$$\sum_{i=1}^{n} c_i \widehat{\alpha}^i = \mathbf{0}, \tag{1.5}$$

where $\mathbf{0}$ is the function that takes all of V to $0 \in \mathbb{R}$. If we evaluate (1.5) at $\mathbf{e}_j$, we get

$$0 = \sum_{i=1}^{n} c_i \widehat{\alpha}^i(\mathbf{e}_j) = \sum_{i=1}^{n} c_i \delta^i{}_j = c_j. \tag{1.6}$$

So $c_j = 0$, and this holds for each $j = 1, 2, \ldots, n$. Therefore, $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ is a linearly independent set that spans V^*, i.e. a basis. ■

Corollary 1.2

The dual space V^* of a finite dimensional vector space has the same dimension as V.

The basis $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ for V^* is said to be dual to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for V.

§1.2 Permutations

Fix a positive integer k. A permutation of the set $A = \{1, 2, \ldots, k\}$ is a bijection $\sigma : A \to A$. The product of two permutations τ and σ is the composition $\tau \circ \sigma : A \to A$. The **cyclic permutation** $(a_1 \ a_2 \ \cdots \ a_r)$ is the permutation σ such that

$$\sigma(a_1) = a_2, \ \sigma(a_2) = a_3, \ \cdots, \ \sigma(a_{r-1}) = a_r, \ \text{and } \sigma(a_r) = 1,$$

leaving all other elements unchanged, i.e. $\sigma(j) = j$ if j is not one of the a_i's. A cyclic permutation $(a_1 \ a_2 \ \cdots \ a_r)$ is also called a **cycle** of length r or an r-cycle. A **transposition** is a permutation of the form $(a \ b)$ that interchanges a and b, leaving all other elements of A fixed.

A permutation $\sigma : A \to A$ can be described by

$$\begin{bmatrix} 1 & 2 & \cdots & k \\ \sigma(1) & \sigma(2) & \cdots & \sigma(k) \end{bmatrix}.$$

We also write it as

$$[1 \ 2 \ \cdots \ k] \overset{\sigma}{\mapsto} [\sigma(1) \ \sigma(2) \ \cdots \ \sigma(k)].$$

Example 1.1. Suppose $\sigma : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ is the permutation given by

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{bmatrix}.$$

In other words, $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 5$, $\sigma(4) = 1$, and $\sigma(5) = 3$.

$$[1 \ 2 \ 3 \ 4 \ 5] \overset{\sigma}{\mapsto} [2 \ 4 \ 5 \ 1 \ 3].$$

Observe that the cyclic permutation $\sigma' = (1 \ 2 \ 4)$ acts as $\sigma'(1) = 2$, $\sigma'(2) = 4$ and $\sigma'(4) = 1$, keeping 3 and 5 unchanged, i.e. $\sigma'(3) = 3$ and $\sigma'(5) = 5$.

$$[1 \ 2 \ 3 \ 4 \ 5] \xmapsto{(1 \ 2 \ 4)} [2 \ 4 \ 3 \ 1 \ 5].$$

Now the transposition $\sigma'' = (3 \ 5)$ acts as $\sigma''(3) = 5$ and $\sigma''(5) = 3$, keeping $1, 2, 4$ unchanged. Therefore,

$$[1 \ 2 \ 3 \ 4 \ 5] \xmapsto{(1 \ 2 \ 4)} [2 \ 4 \ 3 \ 1 \ 5] \xmapsto{(3 \ 5)} [2 \ 4 \ 5 \ 1 \ 3], \qquad [1 \ 2 \ 3 \ 4 \ 5] \xmapsto{(3 \ 5)(1 \ 2 \ 4)} [2 \ 4 \ 5 \ 1 \ 3]$$

so that $\sigma = (3 \ 5)(1 \ 2 \ 4)$.

Let S_k be the group of permutations of the set $\{1, 2, \ldots, k\}$. The order of this group is $k!$. A permutation is *even* or *odd* depending on whether it is the product of an even or an odd number of transpositions. The sign of a permutation σ is 1 if the permutation is even, and -1 otherwise. It is denoted by $\operatorname{sgn} \sigma$. For example, in Example 1.1, $\sigma = (3 \ 5)(1 \ 2 \ 4)$. Note that we can write $(1 \ 2 \ 4)$ as a product of two transpositions:

$$[1 \ 2 \ 3 \ 4 \ 5] \xmapsto{(1 \ 2)} [2 \ 1 \ 3 \ 4 \ 5] \xmapsto{(1 \ 4)} [2 \ 4 \ 3 \ 1 \ 5], \qquad [1 \ 2 \ 3 \ 4 \ 5] \xmapsto{(1 \ 4)(1 \ 2) = (1 \ 2 \ 4)} [2 \ 4 \ 3 \ 1 \ 5]$$

In other words, $\sigma = (3 \ 5)(1 \ 4)(1 \ 2)$. Hence, $\operatorname{sgn} \sigma = -1$. One can easily check that

$$\operatorname{sgn}(\sigma\tau) = (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau). \tag{1.7}$$

So $\operatorname{sgn} : S_k \to \{1, -1\}$ is a group homomorphism.

Example 1.2. Observe that the 5-cycle $(1\ 2\ 3\ 4\ 5)$ can be written as

$$(1\ 2\ 3\ 4\ 5) = (1\ 5)\,(1\ 4)\,(1\ 3)\,(1\ 2)\,.$$

Indeed,

$$[1\ 2\ 3\ 4\ 5] \xmapsto{(1\ 2)} [2\ 1\ 3\ 4\ 5] \xmapsto{(1\ 3)} [2\ 3\ 1\ 4\ 5] \xmapsto{(1\ 4)} [2\ 3\ 4\ 1\ 5] \xmapsto{(1\ 5)} [2\ 3\ 4\ 5\ 1]$$

$$[1\ 2\ 3\ 4\ 5] \xrightarrow{(1\ 2\ 3\ 4\ 5)=(1\ 5)(1\ 4)(1\ 3)(1\ 2)} [2\ 3\ 4\ 5\ 1]$$

Therefore, $\operatorname{sgn}(1\ 2\ 3\ 4\ 5) = 1$.

An **inversion** in a permutation σ is an ordered pair $(\sigma(i), \sigma(j))$ such that $i < j$ but $\sigma(i) > \sigma(j)$. In Example 1.1, $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 5$, $\sigma(4) = 1$, and $\sigma(5) = 3$. So, the inversions in this permutation are

$$(\sigma(1), \sigma(4))\,,\ (\sigma(2), \sigma(4))\,,\ (\sigma(2), \sigma(5))\,,\ (\sigma(3), \sigma(4))\,,\ (\sigma(3), \sigma(5))\,.$$

Hence, there are 5 inversions associated with the permutation σ. There is an efficient way of determining the sign of a permutation.

Proposition 1.3

A permutation is even if and only if it has an even number of inversions.

Proof. Let $\sigma \in S_k$ with n inversions. We shall prove that we can multiply σ by n transpositions and get the identity permutation. This will prove that $\operatorname{sgn}\sigma = (-1)^n$.

Suppose $\sigma(j_1) = 1$. Then for each $i < j_1$, $(\sigma(i), \sigma(j_1))$ is an inversion, and there are $j_1 - 1$ many of them. These are all the inversions with 1 in the second slot of the ordered pair of inversion. If we now multiply σ by the j_1-cycle

$$(\sigma(1)\ \ 1)\,(\sigma(2)\ \ 1)\cdots(\sigma(j_1 - 1)\ \ 1)$$

to the left of σ, the resulting permutation σ_1 would be

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & j_1 & j_1 + 1 & \cdots & k \\ 1 & \sigma(1) & \sigma(2) & \cdots & \sigma(j_1 - 1) & \sigma(j_1 + 1) & \cdots & \sigma(k)\,. \end{bmatrix}$$

This permutation has no inversion with 1 in the second slot of the ordered pair of inversion. Suppose now that $\sigma(j_2) = 2$. Now observe that if $(\sigma_1(i), 2)$ is an inversion in σ_1, then either $(\sigma(i), 2)$ (if $i \geq j_1 + 1$) or $\sigma(i-1), 2$ (if $i \leq j_1 - 1$) is an inversion in σ. Therefore, the number of inversions in σ_1 ending in 2 is precisely the same as the number of inversions in σ ending in 2. So following a similar procedure as above, we can multiply σ_1 by i_2-many transpositions to the left (i_2 is the number of transpositions ending in 2) and get

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & k \\ 1 & 2 & \sigma(1) & \cdots & \sigma(k)\,. \end{bmatrix}$$

We can continue these steps for each $j = 1, 2, \ldots, k$, and the number of transpositions required to move j to its natural position is the same as the number of inversions ending in j. In the end we achieve the identity permutation. Therefore, $\operatorname{sgn}\sigma = (-1)^n$, where n is the number of inversions. ■

§1.3 Multilinear Functions

Definition 1.1. Let V^k be the cartesian product of k-copies of a real vector space V.

$$V^k = \underbrace{V \times V \times \cdots \times V}_{k\text{-copies}}$$

A function $f : V^k \to \mathbb{R}$ is called k-linear if it is linear in each of its k arguments:

$$f(\ldots, a\mathbf{v} + b\mathbf{w}, \ldots) = a\, f(\ldots, \mathbf{v}, \ldots) + b\, f(\ldots, \mathbf{w}, \ldots), \tag{1.8}$$

for $a, b \in \mathbb{R}$ and $\mathbf{v}, \mathbf{w} \in V$.

Instead of 2-linear and 3-linear, it's customary to call "bilinear" and "trilinear", respectively. A k-linear function on V is called a k**-tensor** on V. We will denote the vector space of all k-tensors on V by $L_k(V)$. The vector addition and scalar multiplication of the real vector space $L_k(V)$ is the straightforward pointwise operation.

Example 1.3. The dot product $f(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ on $\mathbb{R}^n$ is bilinear: if $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$ and $\mathbf{w} = \sum_{i=1}^n w^i \mathbf{e}_i$, then

$$f(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^n v^i w^i.$$

Example 1.4. The determinant

$$f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \det \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

viewed as a function of the n column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is n-linear.

Definition 1.2 (Symmetric and alternating function)**.** A k-linear function $f : V^k \to \mathbb{R}$ is **symmetric** if

$$f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right) = f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k), \tag{1.9}$$

for all permutations $\sigma \in S_k$. It is **alternating** if

$$f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right) = (\operatorname{sgn} \sigma)\, f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k), \tag{1.10}$$

for all permutations $\sigma \in S_k$.

The dor product function on $\mathbb{R}^n$ in Theorem 1.3 is symmetric, and the determinant function on $\mathbb{R}^n$ in Theorem 1.4 is alternating.

We are especially interested in the vector space $A_k(V)$ of all alternating k-linear functions on a vector space V, for $k > 0$. The elements of $A_k(V)$ are called alternating k-tensors (also known as k-covectors). We define $A_0(V)$ to be $\mathbb{R}$. The elements of $A_0(V)$ are simply constants, which we call 0-covectors. The elements of $A_1(V)$ are simply covectors, i.e. the elements of V^*.

Permutation action on k-linear functions

If $f \in L_k(V)$ and $\sigma \in S_k$, define $\sigma f \in L_k(V)$ as follows:

$$(\sigma f)(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right). \tag{1.11}$$

Thus, f is symmetric if and only if $f = \sigma f$ for all $\sigma \in S_k$; and f is alternating if and only if $\sigma f = (\operatorname{sgn} \sigma) f$ for all $\sigma \in S_k$. When $k = 1$, S_k only has the identity permutation. In that case, a 1-linear function or simply linear function on V is both symmetric and alternating. In particular,

$$A_1(V) = L_1(V) = V^*.$$

Lemma 1.4

If $\sigma, \tau \in S_k$ and $f \in L_k(V)$, then $\tau(\sigma f) = (\tau\sigma) f$.

Proof. For any $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$,

$$\begin{aligned}
(\tau(\sigma f))(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) &= (\sigma f)\left(\mathbf{v}_{\tau(1)}, \mathbf{v}_{\tau(2)}, \ldots, \mathbf{v}_{\tau(k)}\right) \\
&= (\sigma f)(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k) && [\mathbf{w}_i = \mathbf{v}_{\tau(i)}] \\
&= f\left(\mathbf{w}_{\sigma(1)}, \mathbf{w}_{\sigma(2)}, \ldots, \mathbf{w}_{\sigma(k)}\right) \\
&= f\left(\mathbf{v}_{\tau(\sigma(1))}, \mathbf{v}_{\tau(\sigma(2))}, \ldots, \mathbf{v}_{\tau(\sigma(k))}\right) \\
&= ((\tau\sigma) f)(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k).
\end{aligned}$$

Therefore, $\tau(\sigma f) = (\tau\sigma) f$. ■

Definition 1.3. If G is a group and X is a set, a map

$$\begin{aligned}
G \times X &\to X \\
(g, x) &\mapsto g \cdot x
\end{aligned}$$

is called a **left action** of G on X if

(i) $e \cdot x = x$, where e is the identity element in G and x is any element in X; and

(ii) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$, for all $g_1, g_2 \in G$ and $x \in X$.

Similarly, a **right action** of G on X is a map

$$\begin{aligned}
X \times G &\to X \\
(x, g) &\mapsto x \cdot g
\end{aligned}$$

such that

(i) $x \cdot e = x$, for all $x \in X$; and

(ii) $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$, for all $g_1, g_2 \in G$ and $x \in X$.

Symmetrizing and alternating operators

Given $f \in L_k(V)$, there is a way to make it a symmetric k-linear function $\mathcal{S}f$ from it:

$$(\mathcal{S}f)(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = \sum_{\sigma \in S_k} f\left(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}\right). \tag{1.12}$$

In other words,

$$\mathcal{S}f = \sum_{\sigma \in S_k} \sigma f. \tag{1.13}$$

Similarly, there is a way to make an alternating k-linear function from f:

$$\mathcal{A}f = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, \sigma f. \tag{1.14}$$

Proposition 1.5 (i) The k-linear function $\mathcal{S}f$ is symmetric.

(ii) The k-linear function $\mathcal{A}f$ is alternating.

Proof. (i) Let $\tau \in S_k$. Then

$$\tau(\mathcal{S}f) = \tau\left(\sum_{\sigma \in S_k} \sigma f\right). \tag{1.15}$$

The group action of S_k on $L_k(V)$ is distributive over the vector space addition. Therefore,

$$\tau(\mathcal{S}f) = \sum_{\sigma \in S_k} \tau(\sigma f) = \sum_{\sigma \in S_k} (\tau\sigma) f. \tag{1.16}$$

As σ varies over S_k, $\tau\sigma$ also varies over S_k. Therefore, $\sum_{\sigma \in S_k} (\tau\sigma) f = \mathcal{S}f$. In other words,

$$\tau(\mathcal{S}f) = \mathcal{S}f, \tag{1.17}$$

i.e. $\mathcal{S}f$ is symmetric.

(ii) Let $\tau \in S_k$. Then

$$\tau(\mathcal{A}f) = \tau\left(\sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \sigma f\right) = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \tau(\sigma f) = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)(\tau\sigma) f. \tag{1.18}$$

Since $(\operatorname{sgn}\tau)^2 = 1$,

$$\begin{aligned}\tau(\mathcal{A}f) &= \sum_{\sigma \in S_k} (\operatorname{sgn}\tau)^2 (\operatorname{sgn}\sigma)(\tau\sigma) f \\ &= (\operatorname{sgn}\tau) \sum_{\sigma \in S_k} (\operatorname{sgn}\tau)(\operatorname{sgn}\sigma)(\tau\sigma) f \\ &= (\operatorname{sgn}\tau) \sum_{\sigma \in S_k} (\operatorname{sgn}(\tau\sigma))(\tau\sigma) f. \end{aligned} \tag{1.19}$$

As σ varies over S_k, $\tau\sigma$ also varies over S_k. Therefore, $\sum_{\sigma \in S_k} (\operatorname{sgn}(\tau\sigma))(\tau\sigma) f = \mathcal{A}f$. In other words,

$$\tau(\mathcal{A}f) = \mathcal{A}f, \tag{1.20}$$

i.e. $\mathcal{A}f$ is alternating.

■

Lemma 1.6
If $f \in A_k(V)$, then $\mathcal{A}f = (k!)\, f$.

Proof. Since f is alternating,

$$\mathcal{A}f = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \sigma f = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)^2 f = \sum_{\sigma \in S_k} f = (k!)\, f, \tag{1.21}$$

because the order of S_k is $k!$. ■

§1.4 Tensor Product and Wedge Product

Definition 1.4 (Tensor Product)**.** Let f be a k-linear function and g an l-linear function on a vector space V. Their tensor product $f \otimes g$ is the $(k+l)$-linear function defined by

$$(f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) = f(\mathbf{v}_1, \ldots, \mathbf{v}_k)\, g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}). \tag{1.22}$$

$(k+l)$-linearity of $f \otimes g$ follows from k-linearity of f and l-linearity of g.

Lemma 1.7 (Associativity of Tensor Product)
Let $f \in L_k(V)$, $g \in L_l(V)$ and $h \in L_m(V)$. Then

$$(f \otimes g) \otimes h = f \otimes (g \otimes h) .$$

Proof. For $\mathbf{v}_1, \ldots, \mathbf{v}_{k+l+m}$,

$$\begin{aligned}[(f \otimes g) \otimes h] (\mathbf{v}_1, \ldots, \mathbf{v}_{k+l+m}) &= (f \otimes g) (\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) \, h (\mathbf{v}_{k+l+1}, \ldots, \mathbf{v}_{k+l+m}) \\ &= f (\mathbf{v}_1, \ldots, \mathbf{v}_k) \, g (\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) \, h (\mathbf{v}_{k+l+1}, \ldots, \mathbf{v}_{k+l+m}) . \end{aligned} \tag{1.23}$$

$$\begin{aligned}[f \otimes (g \otimes h)] (\mathbf{v}_1, \ldots, \mathbf{v}_{k+l+m}) &= f (\mathbf{v}_1, \ldots, \mathbf{v}_k) \, (g \otimes h) (\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l+m}) \\ &= f (\mathbf{v}_1, \ldots, \mathbf{v}_k) \, g (\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) \, h (\mathbf{v}_{k+l+1}, \ldots, \mathbf{v}_{k+l+m}) . \end{aligned} \tag{1.24}$$

Therefore, $(f \otimes g) \otimes h = f \otimes (g \otimes h)$, i.e. tensor product is associative. ■

Example 1.5. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{R}^n$, and $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ its dual basis. The Euclidean inner product on $\mathbb{R}^n$ is the bilinear function

$$\langle \, , \, \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

defined by

$$\langle \, , \, \rangle (\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} v^i w^i,$$

for $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$ and $\mathbf{w} = \sum_{i=1}^n w^i \mathbf{e}_i$. We can express $\langle \, , \, \rangle$ in terms of tensor product as follows:

$$\langle \, , \, \rangle (\mathbf{v}, \mathbf{w}) = \sum_{i=1}^{n} v^i w^i = \sum_{i=1}^{n} \widehat{\alpha}^i (\mathbf{v}) \, \widehat{\alpha}^i (\mathbf{w}) = \sum_{i=1}^{n} \left(\widehat{\alpha}^i \otimes \widehat{\alpha}^i \right) (\mathbf{v}, \mathbf{w}) .$$

Since $\mathbf{v}, \mathbf{w}$ are arbitrary,

$$\langle \, , \, \rangle = \sum_{i=1}^{n} \left(\widehat{\alpha}^i \otimes \widehat{\alpha}^i \right) . \tag{1.25}$$

If $f \in A_k(V)$ and $g \in A_l(V)$, then it's not true that $f \otimes g \in A_{k+l}(V)$, in general. We need to construct a product that is also alternating.

Definition 1.5 (Wedge Product)**.** For $f \in A_k(V)$ and $g \in A_l(V)$, the wedge product of f and g is defined as follows:

$$f \wedge g = \frac{1}{k! l!} \mathcal{A} (f \otimes g) . \tag{1.26}$$

Explicitly,

$$\begin{aligned}(f \wedge g) (\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) &= \frac{1}{k! l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) \, \sigma (f \otimes g) (\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) \\ &= \frac{1}{k! l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) (f \otimes g) \left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k+l)} \right) \\ &= \frac{1}{k! l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) \, f \left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k)} \right) g \left(\mathbf{v}_{\sigma(k+1)}, \ldots, \mathbf{v}_{\sigma(k+l)} \right) . \end{aligned} \tag{1.27}$$

When $k=0$, the element $f \in A_0(V)$ is simply a constant $c \in \mathbb{R}$ as discussed earlier. In this case, the wedge product $c \wedge g$ is just scalar multiplication as is evident from (1.27).

$$\begin{aligned}(c \wedge g)(\mathbf{v}_1, \ldots, \mathbf{v}_l) &= \frac{1}{l!} \sum_{\sigma \in S_l} (\operatorname{sgn} \sigma)\, c\, g\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(l)}\right) \\ &= \frac{1}{l!} \sum_{\sigma \in S_l} (\operatorname{sgn} \sigma)\, c\, (\operatorname{sgn} \sigma)\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l) \\ &= \frac{1}{l!} \sum_{\sigma \in S_l} c\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l) \\ &= \frac{1}{l!} l! c\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l) \\ &= c\, g(\mathbf{v}_1, \ldots, \mathbf{v}_l).\end{aligned}$$

Thus $c \wedge g = cg$, for $c \in \mathbb{R}$ and $g \in A_l(V)$.

Example 1.6. For $f \in A_2(V)$ and $g \in A_1(V)$,

$$\begin{aligned}\mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = {} & f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) - f(\mathbf{v}_1, \mathbf{v}_3)\, g(\mathbf{v}_2) - f(\mathbf{v}_2, \mathbf{v}_1)\, g(\mathbf{v}_3) \\ & - f(\mathbf{v}_3, \mathbf{v}_2)\, g(\mathbf{v}_1) + f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1) + f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2).\end{aligned}$$

Among these 6 terms, there are 3 pairs of equal terms due to the alternating nature of f.

$$\begin{aligned}f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) &= -f(\mathbf{v}_2, \mathbf{v}_1)\, g(\mathbf{v}_3), \\ f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2) &= -f(\mathbf{v}_1, \mathbf{v}_3)\, g(\mathbf{v}_2), \\ f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1) &= -f(\mathbf{v}_3, \mathbf{v}_2)\, g(\mathbf{v}_1).\end{aligned}$$

Therefore,

$$\mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 2f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) + 2f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2) + 2f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1). \tag{1.28}$$

Hence,

$$\begin{aligned}(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= \frac{1}{2!1!} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \\ &= f(\mathbf{v}_1, \mathbf{v}_2)\, g(\mathbf{v}_3) + f(\mathbf{v}_3, \mathbf{v}_1)\, g(\mathbf{v}_2) + f(\mathbf{v}_2, \mathbf{v}_3)\, g(\mathbf{v}_1).\end{aligned} \tag{1.29}$$

Example 1.7 (Wedge product of 2 covectors)**.** If $f, g \in A_1(V)$, and $\mathbf{v}_1, \mathbf{v}_2 \in V$, then

$$(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2) = \frac{1}{1!1!} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2).$$

S_2 has 2 elements: the identity element e and $(1\ 2)$. Therefore,

$$(f \wedge g)(\mathbf{v}_1, \mathbf{v}_2) = \mathcal{A}(f \otimes g)(\mathbf{v}_1, \mathbf{v}_2) = f(\mathbf{v}_1)\, g(\mathbf{v}_2) - f(\mathbf{v}_2)\, g(\mathbf{v}_1).$$

Proposition 1.8 (Anticommutativity of wedge product)

The wedge product is anticommutative: if $f \in A_k(V)$ and $g \in A_l(V)$, then

$$f \wedge g = (-1)^{kl}\, g \wedge f.$$

Proof. Define $\tau \in S_{k+l}$ to be the following permutation:

$$\begin{bmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ k+1 & k+2 & \cdots & k+l & 1 & 2 & \cdots & k \end{bmatrix}.$$

In other words,

$$\tau(i) = \begin{cases} k+i & \text{if } 1 \le i \le l, \\ i-l & \text{if } l+1 \le i \le l+k. \end{cases}$$

Then for any $\sigma \in S_{k+l}$,

$$\sigma(j) = \begin{cases} \sigma(\tau(l+j)) & \text{if } 1 \le j \le k, \\ \sigma(\tau(j-k)) & \text{if } k+1 \le j \le k+l. \end{cases} \tag{1.30}$$

Now, for any $\mathbf{v}_1, \ldots, \mathbf{v}_{k+l} \in V$,

$$\begin{aligned} \mathcal{A}(f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) f\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k)}\right) g\left(\mathbf{v}_{\sigma(k+1)}, \ldots, \mathbf{v}_{\sigma(k+l)}\right) \\ &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) f\left(\mathbf{v}_{\sigma(\tau(l+1))}, \ldots, \mathbf{v}_{\sigma(\tau(l+k))}\right) g\left(\mathbf{v}_{\sigma(\tau(1))}, \ldots, \mathbf{v}_{\sigma(\tau(l))}\right) \\ &= (\operatorname{sgn} \tau) \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau) g\left(\mathbf{v}_{\sigma(\tau(1))}, \ldots, \mathbf{v}_{\sigma(\tau(l))}\right) f\left(\mathbf{v}_{\sigma(\tau(l+1))}, \ldots, \mathbf{v}_{\sigma(\tau(l+k))}\right) \\ &= (\operatorname{sgn} \tau) \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma\tau) g\left(\mathbf{v}_{\sigma(\tau(1))}, \ldots, \mathbf{v}_{\sigma(\tau(l))}\right) f\left(\mathbf{v}_{\sigma(\tau(l+1))}, \ldots, \mathbf{v}_{\sigma(\tau(l+k))}\right). \end{aligned}$$

Again, as σ varies over S_{k+l}, $\sigma\tau$ also varies over S_{k+l}. Therefore,

$$\mathcal{A}(f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) = (\operatorname{sgn} \tau) \mathcal{A}(g \otimes f)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}). \tag{1.31}$$

Now, let us evaluate the sign of the permutation τ. Let $(\tau(i), \tau(j))$ be an inversion of τ. Then it's not possible that $1 \le i < j \le l$, or $l+1 \le i < j \le l+k$; because if we have $1 \le i < j \le l$ or $l+1 \le i < j \le l+k$, then $\tau(i) < \tau(j)$. Therefore, i must be in between 1 and l (inclusive), and j must be in between $l+1$ and $l+k$ (inclusive). So there are l options for i, and k options for j. Therefore, τ has kl many inversions. So $\operatorname{sgn} \tau = (-1)^{kl}$. Using (1.31),

$$\mathcal{A}(f \otimes g) = (-1)^{kl} \mathcal{A}(g \otimes f). \tag{1.32}$$

Dividing by $k!l!$, we obtain

$$f \wedge g = (-1)^{kl} g \wedge f. \tag{1.33}$$

■

Corollary 1.9
If f is a k-covector on V, i.e. $f \in A_k(V)$, and k is odd, then $f \wedge f = 0$.

Proof. By anticommutativity of wedge product,

$$f \wedge f = (-1)^{k^2} f \wedge f = -f \wedge f.$$

Therefore, $f \wedge f = 0$. ■

If f is a k-covector and g is an l-covector, i.e. $f \in A_k(V)$ and $g \in A_l(V)$, then we have defined their wedge product to be the $(k+l)$-covector

$$f \wedge g = \frac{1}{k!l!} \mathcal{A}(f \otimes g). \tag{1.34}$$

We have the following lemmas associated with the alternating operator $\mathcal{A}$.

Lemma 1.10

Suppose $f \in L_k(V)$ and $g \in L_l(V)$. Then

(i) $\mathcal{A}(\mathcal{A}(f) \otimes g) = k!\mathcal{A}(f \otimes g)$.

(ii) $\mathcal{A}(f \otimes \mathcal{A}(g)) = l!\mathcal{A}(f \otimes g)$.

Proof. (i) By definition,

$$\begin{aligned}\mathcal{A}(\mathcal{A}(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma)\, \sigma(\mathcal{A}(f) \otimes g) \\ &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma)\, \sigma \left[\sum_{\tau \in S_k} (\operatorname{sgn}\tau)(\tau f) \otimes g\right]. \end{aligned} \tag{1.35}$$

We can view $\tau \in S_k$ as a permutation in the following way: define $\tau' \in S_{k+l}$ as follows

$$\tau'(i) = \begin{cases} \tau(i) & \text{if } i \le k, \\ i & \text{if } i > k. \end{cases} \tag{1.36}$$

Then for $\mathbf{v}_1, \dots, \mathbf{v}_{k+l}$, we have

$$\begin{aligned}[(\tau f) \otimes g](\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) &= (\tau f)(\mathbf{v}_1, \dots, \mathbf{v}_k)\, g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) \\ &= f\left(\mathbf{v}_{\tau(1)}, \dots, \mathbf{v}_{\tau(k)}\right) g(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l}) \\ &= f\left(\mathbf{v}_{\tau'(1)}, \dots, \mathbf{v}_{\tau'(k)}\right) g\left(\mathbf{v}_{\tau'(k+1)}, \dots, \mathbf{v}_{\tau'(k+l)}\right) \\ &= [\tau'(f \otimes g)](\mathbf{v}_1, \dots, \mathbf{v}_{k+l}).\end{aligned}$$

Therefore, $(\tau f) \otimes g = \tau'(f \otimes g)$. Furthermore, $\operatorname{sgn}\tau = \operatorname{sgn}\tau'$ since the inversions $(\tau'(i), \tau'(j))$ occur only when $1 \le i < j \le k$, so that the τ and τ' has the same number of inversions.

Let us abuse notation a bit and denote by S_k the subgroup of permutations in S_{k+l} by keeping the last l arguments fixed. This subgroup of S_{k+l} is indeed isomorphic to S_k, so we will denote both these groups by S_k. Therefore, from (1.35),

$$\begin{aligned}\mathcal{A}(\mathcal{A}(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma)\, \sigma \left[\sum_{\tau' \in S_k \subseteq S_{k+l}} (\operatorname{sgn}\tau')\, \tau'(f \otimes g)\right] \\ &= \sum_{\sigma \in S_{k+l}} \sum_{\tau' \in S_k \subseteq S_{k+l}} (\operatorname{sgn}\sigma)(\operatorname{sgn}\tau')\, \sigma\tau'(f \otimes g) \\ &= \sum_{\tau' \in S_k \subseteq S_{k+l}} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma \operatorname{sgn}\tau')\left((\sigma\tau')(f \otimes g)\right).\end{aligned}$$

For a fixed τ', as σ varies over S_{k+l}, $\sigma\tau'$ also varies over S_{k+l}. Therefore,

$$\mathcal{A}(\mathcal{A}(f) \otimes g) = \sum_{\tau' \in S_k \subseteq S_{k+l}} \mathcal{A}(f \otimes g) = k!\mathcal{A}(f \otimes g). \tag{1.37}$$

(ii) By (1.32),

$$\begin{aligned}\mathcal{A}(f \otimes \mathcal{A}(g)) &= \mathcal{A}\left((-1)^{kl} \mathcal{A}(g) \otimes f\right) \\ &= (-1)^{kl} \mathcal{A}(\mathcal{A}(g) \otimes f) \\ &= (-1)^{kl}\, l!\mathcal{A}(g \otimes f) \\ &= l!\mathcal{A}\left((-1)^{kl} g \otimes f\right) \\ &= l!\mathcal{A}(f \otimes g).\end{aligned} \tag{1.38}$$

■

Proposition 1.11 (Associativity of wedge product)
Let V be a real vector space and f, g, h be alternating multilinear functions on V of degree k, l, m, respectively. Then

$$(f \wedge g) \wedge h = f \wedge (g \wedge h) .$$

Proof. Using the definition of wedge product,

$$\begin{aligned}
(f \wedge g) \wedge h &= \frac{1}{(k+l)!m!} \mathcal{A} \left[(f \wedge g) \otimes h \right] \\
&= \frac{1}{(k+l)!m!} \mathcal{A} \left[\frac{1}{k!l!} \mathcal{A} (f \otimes g) \otimes h \right] \\
&= \frac{1}{(k+l)!k!l!m!} \mathcal{A} \left[\mathcal{A} (f \otimes g) \otimes h \right] \\
&= \frac{(k+l)!}{(k+l)!k!l!m!} \mathcal{A} \left[(f \otimes g) \otimes h \right] \\
&= \frac{1}{k!l!m!} \mathcal{A} \left[(f \otimes g) \otimes h \right] .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
f \wedge (g \wedge h) &= \frac{1}{k! \, (l+m)!} \mathcal{A} \left[f \otimes (g \wedge h) \right] \\
&= \frac{1}{k! \, (l+m)!} \mathcal{A} \left[f \otimes \left(\frac{1}{l!m!} \mathcal{A} (g \otimes h) \right) \right] \\
&= \frac{1}{k! \, (l+m)!l!m!} \mathcal{A} \left[f \otimes \mathcal{A} (g \otimes h) \right] \\
&= \frac{(l+m)!}{k! \, (l+m)!l!m!} \mathcal{A} \left[f \otimes (g \otimes h) \right] \\
&= \frac{1}{k!l!m!} \mathcal{A} \left[f \otimes (g \otimes h) \right] .
\end{aligned}$$

Since tensor product is associative (by Lemma 1.7), we conclude that

$$(f \wedge g) \wedge h = f \wedge (g \wedge h) . \tag{1.39}$$

■

By associativity, we can omit the parenthesis and write univocally $f \wedge g \wedge h$ instead of $(f \wedge g) \wedge h$ or $f \wedge (g \wedge h)$.

Corollary 1.12
Under the hypothesis of Proposition 1.11,

$$f \wedge g \wedge h = \frac{1}{k!l!m!} \mathcal{A} \left[f \otimes g \otimes h \right] . \tag{1.40}$$

This easily generalizes to an arbitrary number of factors: if $f_i \in A_{d_i}(V)$ for $i = 1, 2, \dots, r$, i.e. f_i is an alternating d_i-linear function on V, then

$$f_1 \wedge \cdots \wedge f_r = \frac{1}{d_1! \cdots d_r!} \mathcal{A} (f_1 \otimes \cdots \otimes f_r) . \tag{1.41}$$

Proposition 1.13

Let $\widehat{\alpha}^1, \widehat{\alpha}^2, \ldots, \widehat{\alpha}^k$ be linear functions on a real vector space V (i.e. $\widehat{\alpha}^i : V \to \mathbb{R}$) and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$. Then

$$\begin{aligned}\left(\widehat{\alpha}^1 \wedge \cdots \wedge \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k) &= \det\left[\widehat{\alpha}^i(\mathbf{v}_j)\right] \\ &= \det\begin{bmatrix} \widehat{\alpha}^1(\mathbf{v}_1) & \widehat{\alpha}^1(\mathbf{v}_2) & \cdots & \widehat{\alpha}^1(\mathbf{v}_k) \\ \widehat{\alpha}^2(\mathbf{v}_1) & \widehat{\alpha}^2(\mathbf{v}_2) & \cdots & \widehat{\alpha}^2(\mathbf{v}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\alpha}^k(\mathbf{v}_1) & \widehat{\alpha}^k(\mathbf{v}_2) & \cdots & \widehat{\alpha}^k(\mathbf{v}_k) \end{bmatrix}.\end{aligned}$$

Proof. By 1.41,

$$\left(\widehat{\alpha}^1 \wedge \cdots \wedge \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \mathcal{A}\left(\widehat{\alpha}^1 \otimes \cdots \otimes \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k).$$

By the definition of the action of alternating operator,

$$\mathcal{A}\left(\widehat{\alpha}^1 \otimes \cdots \otimes \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \widehat{\alpha}^1\left(\mathbf{v}_{\sigma(1)}\right) \cdots \widehat{\alpha}^k\left(\mathbf{v}_{\sigma(k)}\right). \tag{1.42}$$

By the definition of determinant of a $k \times k$ matrix $A = [a_{ij}]$,

$$\det A = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{k\sigma(k)}. \tag{1.43}$$

Using (1.43) in (1.42), we get

$$\mathcal{A}\left(\widehat{\alpha}^1 \otimes \cdots \otimes \widehat{\alpha}^k\right)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \det\left[\widehat{\alpha}^i(\mathbf{v}_j)\right]. \tag{1.44}$$

■

§1.5 A Basis for $A_k(V)$

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for a real vector space V, and let $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ be the dual basis for V^*. Introduce the multi-index notation

$$I = (i_1, i_2, \ldots, i_k)$$

and write $\mathbf{e}_I$ for $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_k})$ and $\widehat{\alpha}^I$ for $\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k}$.

A k-linear function f on V is completely determined by its values on all k-tuples $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_k})$. If f is alternating, then f is completely determined by its values on all k-tuples $(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_k})$ with

$$1 \le i_1 < i_2 < \cdots < i_k \le n.$$

In other words, it's sufficient to consider $\mathbf{e}_I$ with I in ascending order.

Lemma 1.14

Suppose I and J are ascending multi-indices of length k. Then

$$\widehat{\alpha}^I(\mathbf{e}_J) = \delta^I{}_J := \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases}$$

Proof. Suppose $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$. Using (1.42), we get

$$\begin{aligned}\widehat{\alpha}^I(\mathbf{e}_J) &= \left(\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k}\right)(\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \ldots, \mathbf{e}_{j_k}) \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \widehat{\alpha}^{i_1}\left(\mathbf{e}_{j_{\sigma(1)}}\right) \cdots \widehat{\alpha}^{i_k}\left(\mathbf{e}_{j_{\sigma(k)}}\right) \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \delta^{i_1}{}_{j_{\sigma(1)}} \cdots \delta^{i_k}{}_{j_{\sigma(k)}}.\end{aligned} \tag{1.45}$$

The terms in the sum (1.45) contribute $\operatorname{sgn}\sigma$ if and only if

$$(i_1, i_2, \ldots, i_k) = \left(j_{\sigma(1)}, j_{\sigma(2)}, \ldots, j_{\sigma(k)}\right);$$

otherwise they contribute 0 to the sum. Both I and J are ascending multi-indices. Permuting the elements of J no longer gives an ascending multi-index (unless the permutation σ is the identity permutation). Therefore, in (1.45), all the summands corresponding to σ being a non-identity permutation contribute 0.

$$\widehat{\alpha}^I(\mathbf{e}_J) = \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, \delta^{i_1}{}_{j_{\sigma(1)}} \cdots \delta^{i_k}{}_{j_{\sigma(k)}} = \delta^{i_1}{}_{j_1} \cdots \delta^{i_k}{}_{j_k} = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases} \tag{1.46}$$

■

Proposition 1.15

The alternating k-linear functions $\widehat{\alpha}^I$, $I = (i_1, \ldots, i_k)$, with $1 \le i_1 < \cdots < i_k \le n$ form a basis for the space $A_k(V)$ of alternating k-linear functions on V.

Proof. Let us first show linear independence. Suppose

$$\sum_I c_I \widehat{\alpha}^I = \mathbf{0}, \tag{1.47}$$

$c_I \in \mathbb{R}$ with I running over ascending multi-indices of length k. Applying $\mathbf{e}_J$ to both sides, we get

$$0 = \sum_I c_I \widehat{\alpha}^I(\mathbf{e}_J) = \sum_I c_I \delta^I{}_J = c_J. \tag{1.48}$$

Therefore, $\left\{\widehat{\alpha}^I \mid I \text{ is ascending multi-index of length } k\right\}$ is a linearly independent set. Now let us prove that this set spans $A_k(V)$. Let $f \in A_k(V)$. We claim that

$$f = \sum_I f(\mathbf{e}_I)\, \widehat{\alpha}^I. \tag{1.49}$$

Let $g = \sum_I f(\mathbf{e}_I)\, \widehat{\alpha}^I$. We need to prove that $f = g$. By k-linearity and alternating property, if two k-covectors agree on all $\mathbf{e}_J$ where J is an ascending multi-index, then they are equal. Now,

$$g(\mathbf{e}_J) = \sum_I f(\mathbf{e}_I)\, \widehat{\alpha}^I(\mathbf{e}_J) = \sum_I f(\mathbf{e}_I)\, \delta^I{}_J = f(\mathbf{e}_J). \tag{1.50}$$

Therefore, $f = g = \sum_I f(\mathbf{e}_I)\, \widehat{\alpha}^I$. ■

Corollary 1.16

If the vector space V has dimension n, then the vector space $A_k(V)$ of k-covectors on V has dimension $\binom{n}{k}$.

Proof. An ascending multi-index $I = (i_1, i_2, \ldots, i_k)$, $1 \le i_1 < i_2 < \cdots < i_k \le n$ is obtained by choosing a k-element subset of $\{1, 2, \ldots, n\}$. This can be done in $\binom{n}{k}$ ways. ■

Corollary 1.17

If $k > \dim V$, then $A_k(V) = 0$.

Proof. If $k > \dim V = n$, then in the expression

$$\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k}$$

with each $i \in \{1, 2, \ldots, n\}$, there must be a repeated i_j's, say $\widehat{\alpha}^r$. Then $\widehat{\alpha}^r \wedge \widehat{\alpha}^r$ arises in the expression $\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k}$. But $\widehat{\alpha}^r \wedge \widehat{\alpha}^r = 0$ by Corollary 1.9. Hence, $\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k} = 0$. Therefore, the basis set of $A_k(V)$ is empty, meaning $A_k(V) = 0$. ■

2 Differential Forms on $\mathbb{R}^n$

Given an open set $U \subseteq \mathbb{R}^n$ and $p \in U$, T_pU is the set of tangent vectors at $p \in U$ is identified with the point derivations of C_p^∞ (germs of smooth functions at p), i.e. a tangent vector $X_p \in T_pU$ is a map $X_p : C_p^\infty \to \mathbb{R}$ such that X_p is $\mathbb{R}$-linear:

$$X_p\left(\alpha f + g\right) = \alpha\left(X_p f\right) + X_p g; \tag{2.1}$$

and satisfies the Leibniz condition:

$$X_p\left(fg\right) = \left(X_p f\right) g(p) + f(p)\left(X_p g\right).$$

In contrast to the notion of point derivation, there is this notion of derivation of aa algebra. If X is a C^∞ vector field on an open subset $U \subseteq \mathbb{R}^n$, i.e. $X \in \mathfrak{X}(U)$, and f is a C^∞ function on U, i.e. $f \in C^\infty(U)$, then $Xf \in C^\infty(U)$ defined by

$$\left(Xf\right)(p) = X_p f.$$

Remember that f in (2.1) and (2) is a representative of an equivalence class, the equivalence class of germs of C^∞ functions at $p \in U$. These equivalence classes constitute $C_p^\infty(U)$. It is of course an $\mathbb{R}$-algebra. While in (2), $f \in C^\infty(U)$, the algebra of C^∞ functions on U with no reference of p whatsoever.

From the discussion above, a C^∞ vector field X gives rise to an $\mathbb{R}$-linear map $C^\infty(U) \to C^\infty(U)$ by $f \mapsto Xf$ that additionally has to satisfy the following Leibniz condition:

$$X\left(fg\right) = \left(Xf\right)g + f\left(Xg\right). \tag{2.2}$$

Note that a derivation at p is not a derivation of the algebra C_p^∞. A derivation at p is a map from $C_p^\infty \to \mathbb{R}$ that satisfies (2), while a derivation of the algebra C_p^∞ is supposed to be a map from C_p^∞ to itself obeying Leibniz condition.

§2.1 1-form

From any C^∞ function $f : U \to \mathbb{R}$, one can construct a 1-form (dual notion of C^∞ vector field) $\mathrm{d}f$, the restriction of which to a given point $p \in U$ yields a covector $\left(\mathrm{d}f\right)_p \in T_p^*U$, the dual space of T_pU, in the following way:

$$\left(\mathrm{d}f\right)_p\left(X_p\right) = X_p f. \tag{2.3}$$

Proposition 2.1

If $x^1, x^2, \ldots, x^n$ are the standard coordinates on $\mathbb{R}^n$, then at each point $p \in \mathbb{R}^n$,

$$\left\{\left(\mathrm{d}x^1\right)_p, \left(\mathrm{d}x^2\right)_p, \ldots, \left(\mathrm{d}x^n\right)_p\right\}$$

is the basis for the cotangent space $T_p^*\mathbb{R}^n$ dual to the basis $\left\{\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p\right\}$ for the tangent space $T_p\mathbb{R}^n$.

Proof. $\left(\mathrm{d}x^i\right)_p : T_p^*\mathbb{R}^n \to \mathbb{R}$ is a linear map for each i. Now,

$$\left(\mathrm{d}x^i\right)_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \left.\frac{\partial}{\partial x^j}\right|_p\left(x^i\right) = \delta^i{}_j. \tag{2.4}$$

Therefore, $\left\{(\mathrm{d}x^1)_p, (\mathrm{d}x^2)_p, \ldots, (\mathrm{d}x^n)_p\right\}$ is the basis of $T_p^*\mathbb{R}^n$ dual to the basis $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ for $T_p\mathbb{R}^n$. ■

If ω is a 1-form on an open subset $U \subseteq \mathbb{R}^n$, then by Proposition 2.1, there is a linear combination

$$\omega_p = \sum_{i=1}^{n} a_i(p) \left(\mathrm{d}x^i\right)_p, \tag{2.5}$$

for some $a_i(p) \in \mathbb{R}$. As p varies over U, the coefficients a_i become functions on U, and we may write

$$\omega = \sum_{i=1}^{n} a_i \mathrm{d}x^i. \tag{2.6}$$

The 1-form ω is said to be C^∞ on U if the coefficient functions a_i are all C^∞ functions on U.

Proposition 2.2 (The differential in terms of coordinates)
If $f : U \to \mathbb{R}$ is a C^∞ function on an open set $U \subseteq \mathbb{R}^n$, then

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i.$$

Proof. By Proposition 2.1, at each point $p \in U$,

$$(\mathrm{d}f)_p = \sum_{i=1}^{n} a_i(p) \left(\mathrm{d}x^i\right)_p, \tag{2.7}$$

for some constants $a_i(p)$ depending on p. Thus

$$\mathrm{d}f = \sum_{i=1}^{n} a_i \; \mathrm{d}x^i, \tag{2.8}$$

for some functions a_i on U. To evaluate a_j, apply both sides of (2.8) to the coordinate vector field $\frac{\partial}{\partial x^j}$:

$$\mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^{n} a_i \; \mathrm{d}x^i \left(\frac{\partial}{\partial x^j}\right) = \sum_{i=1}^{n} a_i {\delta^i}_j = a_j. \tag{2.9}$$

On the other hand, using $(\mathrm{d}f)_p (X_p) = X_p f = (Xf)(p)$, we get $(\mathrm{d}f)(X) = Xf$. So

$$\mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}. \tag{2.10}$$

Therefore, $a_j = \frac{\partial f}{\partial x^j}$. Hence,

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \; \mathrm{d}x^i. \tag{2.11}$$

■

(2.11) tells us that $\mathrm{d}f$ will be a C^∞ 1-form if $\frac{\partial f}{\partial x^i}$ is C^∞ on U. Hence, it is sufficient to have f as a C^∞ function on U in order to have $\mathrm{d}f$ as a C^∞ 1-form.

§2.2 Differential k-forms

A differential form ω of degree k (or a k-form) on an open subset $U \subseteq \mathbb{R}^n$ is a map that assigns to each point $p \in U$, an alternating k-linear function on the tangent space $T_p\mathbb{R}^n$, i.e.

$$\omega_p \in A_k\left(T_p\mathbb{R}^n\right).$$

By Proposition 1.15, a basis for $A_k\left(T_p\mathbb{R}^n\right)$ is

$$\left(\mathrm{d}x^I\right)_p = \left(\mathrm{d}x^{i_1}\right)_p \wedge \left(\mathrm{d}x^{i_2}\right)_p \wedge \cdots \wedge \left(\mathrm{d}x^{i_k}\right)_p,$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Therefore, at each point $p \in U$, ω_p is a linear combination

$$\omega_p = \sum_I a_I(p) \left(\mathrm{d}x^I\right)_p, \tag{2.12}$$

and a k-form ω on U is a linear combination

$$\omega = \sum_I a_I \ \mathrm{d}x^I, \tag{2.13}$$

with function coefficients $a_I : U \to \mathbb{R}$. We say that a k-form ω is **smooth** on U if all the coeffieicnts a_I are C^∞ functions on U.

Denote by $\Omega^k(U)$ the vector space of C^∞ k-forms on U. A 0-form on U assigns to each point $p \in U$ an element of $A_0\left(T_p\mathbb{R}^n\right) = \mathbb{R}$. Thus a 0-form on U is simply a real-valued function on U, and $\Omega^0\left(U\right) = C^\infty(U)$.

Since one can multiply a C^∞ k-form by a C^∞ function on U from the left, the set $\Omega^k(U)$ of C^∞ k-forms on U is both a real vector space and a $C^\infty(U)$-module. With the wedge product as multiplication, the direct sum

$$\Omega^*(U) = \bigoplus_{k=0}^{n} \Omega^k(U)$$

becomes an algebra over $\mathbb{R}$ as well as a module over $C^\infty(U)$. As an algebra, it is anticommutative and associative.

Example 2.1. Let x, y, z be the coordinates on $\mathbb{R}^3$. The C^∞ 1-forms on $\mathbb{R}^3$ are

$$P\left(x,y,z\right) \ \mathrm{d}x + Q\left(x,y,z\right) \ \mathrm{d}y + R\left(x,y,z\right) \ \mathrm{d}z,$$

where P, Q, R range over all C^∞ functions on $\mathbb{R}^3$. The C^∞ 2-forms are

$$A\left(x,y,z\right) \ \mathrm{d}y \wedge \mathrm{d}z + B\left(x,y,z\right) \ \mathrm{d}x \wedge \mathrm{d}z + C\left(x,y,z\right) \ \mathrm{d}x \wedge \mathrm{d}y;$$

and the C^∞ 1-forms are

$$a\left(x,y,z\right) \ \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

Example 2.2 (A basis for 3-covectors)**.** Let x^1, x^2, x^3, x^4 be the standard coordinates on $\mathbb{R}^4$, and $p \in \mathbb{R}^4$. A basis for $A_3\left(T_p\mathbb{R}^4\right)$ is

$$\left\{\left(\mathrm{d}x^1\right)_p \wedge \left(\mathrm{d}x^2\right)_p \wedge \left(\mathrm{d}x^3\right)_p, \ \left(\mathrm{d}x^1\right)_p \wedge \left(\mathrm{d}x^2\right)_p \wedge \left(\mathrm{d}x^4\right)_p, \right.$$
$$\left. \left(\mathrm{d}x^1\right)_p \wedge \left(\mathrm{d}x^3\right)_p \wedge \left(\mathrm{d}x^4\right)_p, \left(\mathrm{d}x^2\right)_p \wedge \left(\mathrm{d}x^3\right)_p \wedge \left(\mathrm{d}x^4\right)_p\right\}.$$

So $\dim\left(A_3\left(T_p\mathbb{R}^n\right)\right) = 4$.

§2.3 Exterior Derivative

Before defining exterior derivative of a C^∞ k-form on an open subset $U \subseteq \mathbb{R}^n$, we first define it on 0-forms. The exterior derivative of a C^∞ function $f \in C^\infty(U)$ is its differential:

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i \in \Omega^1(U).$$

Definition 2.1 (Exterior Derivative). If $\omega = \sum_I a_I \, \mathrm{d}x^I \in \omega^K(U)$, then its exterior derivative is defined as follows:

$$\mathrm{d}\omega = \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I = \sum_I \left(\sum_{j=1}^{n} \frac{\partial a_I}{\partial x^j} \, \mathrm{d}x^j \right) \wedge \mathrm{d}x^I \in \Omega^{k+1}(U). \tag{2.14}$$

Example 2.3. Let ω be the 1 form $f\mathrm{d}x + g\mathrm{d}y$ on $\mathbb{R}^2$, where f and g are C^∞ functions on $\mathbb{R}^2$. Let us write $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$. Then

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}f \wedge \mathrm{d}x + \mathrm{d}g \wedge \mathrm{d}y \\ &= (f_x \mathrm{d}x + f_y \mathrm{d}y) \wedge \mathrm{d}x + (g_x \mathrm{d}x + g_y \mathrm{d}y) \wedge \mathrm{d}y \\ &= -f_y \, \mathrm{d}x \wedge \mathrm{d}y + g_x \, \mathrm{d}x \wedge \mathrm{d}y \\ &= (g_x - f_y) \, \mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

Definition 2.2 (Graded Algebra). An algebra A over a field $\mathbb{K}$ is said to be **graded** if it can be written as a direct sum

$$A = \bigoplus_{k=0}^{\infty} A^k$$

of vector spaces over $\mathbb{K}$ so that the multiplication map sends $A^k \times A^l$ to A^{k+l}.

The notation $A = \bigoplus_{k=0}^{\infty} A^k$ means that each element of A is uniquely a **finite sum**

$$a = a_{i_1} + a_{i_2} + \cdots + a_{i_m},$$

where $a_{i_j} \in A^{i_j}$.

Example 2.4. The polynomial algebra

$$\mathbb{R}[x, y] = \bigoplus_{k=0}^{\infty} A^k$$

with A^k being the vector space of homogenous polynomials of degree k in x and y. Observe that the 0 polynomial is trivially homogenous of any degree, and hence belongs to A^k for all $k \geq 0$. Multiplication of degree k homogenous polynomial with a degree l homogenous polynomial in x and y will result in a homogenous polynomial of degree $k + l$ in x and y.

Example 2.5. The algebra $\Omega^*(U)$ of C^∞ differential forms on U is also graded by the degree of differential forms. Each $\Omega^k(U)$ is a vector space. Multiplication of differential forms is defined by wedge product between them. The wedge product of a degree k differential form on U with a degree l differential form results in a degree $k + l$ differential form.

Definition 2.3 (Anti-derivation)**.** Let $A = \bigoplus_{k=0}^{\infty} A^k$ be a graded algebra over a field $\mathbb{K}$. An **anti-derivation** of the graded algebra A is a $\mathbb{K}$-linear map $D : A \to A$ such that for $\omega \in A^k$ and $\tau \in A^l$, one has

$$D(\omega\tau) = (D\omega)\tau + (-1)^k \omega (D\tau). \tag{2.15}$$

If the antiderivation D sends $\omega \in A^k$ to $D\omega \in A^{k+m}$, we say that it is an antiderivation of degree m.

Proposition 2.3 (i) The exterior derivative $\mathrm{d} : \Omega^*(U) \to \Omega^*(U)$ is an antiderivation of degree 1:

$$\mathrm{d}(\omega \wedge \tau) = (\mathrm{d}\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge \mathrm{d}\tau. \tag{2.16}$$

(ii) $\mathrm{d}^2 = 0$.

(iii) If $f \in \Omega^0(U) = C^\infty(U)$ and $X \in \mathfrak{X}(U)$ (the space of C^∞ vector fields), then $(\mathrm{d}f)(X) = Xf$.

Proof. (i) Since the exterior derivative operator $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$ is linear, it suffices to check the equality (2.16) for $\omega = f\,\mathrm{d}x^I$ and $\tau = g\,\mathrm{d}x^J$ with $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_l)$ being strictly ascending multi-indices.

$$\begin{aligned}
\mathrm{d}(\omega \wedge \tau) &= \mathrm{d}\left(fg\mathrm{d}x^I \wedge \mathrm{d}x^J\right) \\
&= \sum_{i=1}^{n} \frac{\partial (fg)}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J \\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \cdot g\mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J + \sum_{i=1}^{n} f \cdot \frac{\partial g}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J \\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^I \wedge g\mathrm{d}x^J + \sum_{i=1}^{n} \frac{\partial g}{\partial x^i} \mathrm{d}x^i \wedge f\,\mathrm{d}x^I \wedge \mathrm{d}x^J.
\end{aligned} \tag{2.17}$$

Now, in the second sum in (2.17), one has to push $\frac{\partial g}{\partial x^i}\mathrm{d}x^i$ through the k-fold wedge product $\mathrm{d}x^I$ and hence in the process picks out a sign $(-1)^k$. Therefore,

$$\sum_{i=1}^{n} \frac{\partial g}{\partial x^i} \mathrm{d}x^i \wedge f\,\mathrm{d}x^I \wedge \mathrm{d}x^J = (-1)^k f\,\mathrm{d}x^I \wedge \frac{\partial g}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^J. \tag{2.18}$$

Now, observe that

$$\mathrm{d}\omega = \mathrm{d}\left(f\,\mathrm{d}x^I\right) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^I, \text{ and} \tag{2.19}$$

$$\mathrm{d}\tau = \mathrm{d}\left(g\,\mathrm{d}x^J\right) \sum_{i=1}^{n} \frac{\partial g}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^J. \tag{2.20}$$

Therefore,

$$\mathrm{d}(\omega \wedge \tau) = \mathrm{d}\omega \wedge \tau + (-1)^k \omega \wedge \mathrm{d}\tau. \tag{2.21}$$

(ii) Again, by $\mathbb{R}$-linearity of d, it suffices to show that $\mathrm{d}^2\omega = 0$ for $\omega = f\mathrm{d}x^I$.

$$\begin{aligned}
\mathrm{d}^2\left(f\mathrm{d}x^I\right) &= \mathrm{d}\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^I\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x^j \partial x^i} \mathrm{d}x^j \wedge \mathrm{d}x^i \wedge \mathrm{d}x^I.
\end{aligned} \tag{2.22}$$

If $i = j$, then $\mathrm{d}x^j \wedge \mathrm{d}x^i = 0$. If $i \neq j$, then $\frac{\partial^2 f}{\partial x^j \partial x^i}$ is symmetric in i and j, but $\mathrm{d}x^j \wedge \mathrm{d}x^i$ is alternating in i and j. Therefore, the terms with $i \neq j$ pair up and cancel out.

(iii) Let $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$. Then

$$\begin{aligned}(\mathrm{d}f)(X) &= \left(\sum_{j=1}^{n} \frac{\partial f}{\partial x^j} \mathrm{d}x^j\right)\left(\sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}\right) \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n} a^i \frac{\partial f}{\partial x^j} \mathrm{d}x^j \left(\frac{\partial}{\partial x^i}\right) \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n} a^i \frac{\partial f}{\partial x^j} \delta^j{}_i \\ &= \sum_{i=1}^{n} a^i \frac{\partial f}{\partial x^i} = Xf. \end{aligned} \tag{2.23}$$

■

Proposition 2.4 (Characterization of exterior derivative)

The 3 properties of Proposition 2.3 uniquely characterize exterior derivative on an open set $U \subseteq \mathbb{R}^n$. In other words, if $D : \Omega^*(U) \to \Omega^*(U)$ is an antiderivation of degree 1 such that $D^2 = 0$ and for $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, $(Df)(X) = Xf$, then $D = \mathrm{d}$.

Proof. Since every k-form on U is a sum of terms such as $f\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$, by linearity of d, it suffices to show that $D = \mathrm{d}$ on a k-form of this type. Applying property (iii) for $f = x^i$, one has

$$Dx^i (X) = X\left(x^i\right).$$

Writing $X = \sum_{j=1}^{n} a^j \frac{\partial}{\partial x^j}$, we get

$$Dx^i\left(\sum_{j=1}^{n} a^j \frac{\partial}{\partial x^j}\right) = \sum_{j=1}^{n} a^j \frac{\partial}{\partial x^j}\left(x^i\right) = a^i = \mathrm{d}x^i\left(\sum_{j=1}^{n} a^j \frac{\partial}{\partial x^j}\right).$$

Therefore,

$$Dx^i = \mathrm{d}x^i. \tag{2.24}$$

Now,

$$\begin{aligned} D\left(f\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) &= D\left(fDx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) \\ &= Df \wedge \left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) + (-1)^0 fD\left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right). \end{aligned} \tag{2.25}$$

Now, since $\mathrm{d}f(X) = Xf = Df(X)$ for any $X \in \mathfrak{X}(U)$, $\mathrm{d}f = Df$. Furthermore, $D\left(Dx^{i_1}\right) = 0$, and

$$\begin{aligned} D\left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) &= D^2 x^{i_1} \wedge Dx^{i_2} \wedge \cdots \wedge Dx^{i_k} - Dx^{i_1} \wedge D\left(Dx^{i_2} \wedge \cdots \wedge Dx^{i_k}\right) \\ &= -Dx^{i_1} \wedge D\left(Dx^{i_2} \wedge \cdots \wedge Dx^{i_k}\right). \end{aligned} \tag{2.26}$$

Therefore, by induction on k,

$$D\left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) = 0. \tag{2.27}$$

Hence, from (2.25),

$$\begin{aligned} D\left(f\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) &= Df \wedge \left(Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}\right) \\ &= \mathrm{d}f \wedge \left(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) \\ &= \mathrm{d}\left(f\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right). \end{aligned} \tag{2.28}$$

So $D = \mathrm{d}$ on $\Omega^*(U)$. ■

Closed Forms and Exact Forms

A k-form ω on U is **closed** if $\mathrm{d}\omega = 0$; it's **exact** if there is a $(k-1)$-form τ on U such that $\omega = \mathrm{d}\tau$. Since $\mathrm{d}^2 = 0$, every exact form is closed. But in general, a closed form may fail to be exact. We will see how non-exact closed forms capture the geometry of a manifold when we do de Rham cohomology on a manifold.

Example 2.6. Define a 1-form ω on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ by

$$\omega = \frac{1}{x^2+y^2}\left(-y\mathrm{d}x + x\mathrm{d}y\right). \tag{2.29}$$

Then ω is closed.

A collection of vector spaces $\left\{V^k\right\}_{k=0}^{\infty}$ with linear maps $d_k : V^k \to V^{k+1}$ such that $d_{k+1} \circ d_k = 0$ is called a **differential complex** or a **cochain complex**. For aany open set $U \subseteq \mathbb{R}^n$, the exterior derivative d makes the vector space $\Omega^*(U)$ of C^∞ forms on U into a cochain complex, called the **de Rham complex** on U:

$$\Omega^0(U) \xrightarrow{\mathrm{d}} \Omega^1(U) \xrightarrow{\mathrm{d}} \Omega^2(U) \xrightarrow{\mathrm{d}} \cdots$$

The closed forms are precisely the elements of the kernel of d and the exact forms are the elements of the image of d. In the language of cohomology, d is also called the coboundary operator.

§2.4 Applications to Vector Calculus

The theory of differential forms unifies many theorems in vector calculus on $\mathbb{R}^3$. A vector valued function on $\mathbb{R}^3$ is the same as a vector field. Recall the 3 operators on scalar and vector-valued functions on $\mathbb{R}^3$.

$$\{\text{scalar function}\} \xrightarrow{\text{grad}} \{\text{vector function}\} \xrightarrow{\text{curl}} \{\text{vector function}\} \xrightarrow{\text{div}} \{\text{scalar function}\}.$$

Let f be a scalar function and $\begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ be a vector field on $\mathbb{R}^3$, where each of P, Q, R is a scalar function on $\mathbb{R}^3$. Then

$$\begin{aligned} \operatorname{grad} f &= \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}, \\ \operatorname{curl} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} &= \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix}, \\ \operatorname{div} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} &= P_x + Q_y + R_z. \end{aligned} \tag{2.30}$$

Then one has the following results.

Proposition 2.5

$$\operatorname{curl}(\operatorname{grad} f) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2.31}$$

Proposition 2.6

$$\operatorname{div}\left(\operatorname{curl}\begin{bmatrix}P\\Q\\R\end{bmatrix}\right)=0. \tag{2.32}$$

Proposition 2.7

On $\mathbb{R}^3$, a vector field $\mathbf{F}(x,y,z)$ is the gradient of some scalar function if and only if $\operatorname{curl}\mathbf{F}=\mathbf{0}$.

A 1-form on $\mathbb{R}^3$ can be written as

$$P(x,y,z)\,\mathrm{d}x+Q(x,y,z)\,\mathrm{d}y+R(x,y,z)\,\mathrm{d}z.$$

This 1-form on $\mathbb{R}^3$ can be identified with the vector field $\begin{bmatrix}P\\Q\\R\end{bmatrix}$.

Similarly, the 2-forms on $\mathbb{R}^3$ given by

$$A(x,y,z)\,\mathrm{d}y\wedge\mathrm{d}z+B(x,y,z)\,\mathrm{d}z\wedge\mathrm{d}x+C(x,y,z)\,\mathrm{d}x\wedge\mathrm{d}y$$

can be identified with the vector field $\begin{bmatrix}A\\B\\C\end{bmatrix}$ on $\mathbb{R}^3$.

In terms of these identifications, the exterior derivative of a 0-form f (scalar function) is

$$\mathrm{d}f=\frac{\partial f}{\partial x}\mathrm{d}x+\frac{\partial f}{\partial y}\mathrm{d}y+\frac{\partial f}{\partial z}\mathrm{d}z,$$

which can be identified with the vector field

$$\begin{bmatrix}\frac{\partial f}{\partial x}\\\frac{\partial f}{\partial y}\\\frac{\partial f}{\partial z}\end{bmatrix}=\operatorname{grad}f.$$

The exterior derivative of a 1-form on $\mathbb{R}^3$ is

$$\begin{aligned}&\mathrm{d}(P\mathrm{d}x+Q\mathrm{d}y+R\mathrm{d}z)\\=&\frac{\partial P}{\partial y}\mathrm{d}y\wedge\mathrm{d}x+\frac{\partial P}{\partial z}\mathrm{d}z\wedge\mathrm{d}x+\frac{\partial Q}{\partial x}\mathrm{d}x\wedge\mathrm{d}y+\frac{\partial Q}{\partial z}\mathrm{d}z\wedge\mathrm{d}y+\frac{\partial R}{\partial x}\mathrm{d}x\wedge\mathrm{d}z+\frac{\partial R}{\partial y}\mathrm{d}y\wedge\mathrm{d}z\\=&(R_y-Q_z)\,\mathrm{d}y\wedge\mathrm{d}z+(P_z-R_x)\,\mathrm{d}z\wedge\mathrm{d}x+(Q_x-P_y)\,\mathrm{d}x\wedge\mathrm{d}y,\end{aligned}$$

which corresponds to

$$\begin{bmatrix}R_y-Q_z\\P_z-R_x\\Q_x-P_y\end{bmatrix}=\operatorname{curl}\begin{bmatrix}P\\Q\\R\end{bmatrix}.$$

The exterior derivative of a 2-form is

$$\begin{aligned}&\mathrm{d}(A\mathrm{d}y\wedge\mathrm{d}z+B\mathrm{d}z\wedge\mathrm{d}x+C\mathrm{d}x\wedge\mathrm{d}y)\\=&A_x\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z+B_y\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}x+C_z\mathrm{d}z\wedge\mathrm{d}x\wedge\mathrm{d}y\\=&(A_x+B_y+C_z)\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z,\end{aligned}$$

which corresponds to

$$A_x+B_y+C_z=\operatorname{div}\begin{bmatrix}A\\B\\C\end{bmatrix}.$$

In summary, exterior derivative d on 0-forms is identified with **gradient**; exterior derivative d on 1-forms is identified with **curl**; exterior derivative d on 2-forms is identified with **divergence**. Using de Rham complex on U:

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \Omega^3(U).$$

Using vector calculus language,

$$C^\infty(U) \xrightarrow{\text{grad}} \mathfrak{X}(U) \xrightarrow{\text{curl}} \mathfrak{X}(U) \xrightarrow{\text{div}} C^\infty(U).$$

Remark 2.1. Proposition 2.5 and Proposition 2.6 express the property $d^2 = 0$ of exterior derivative. A vector field $\begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ is the gradient of a C^∞ function f if and only if the corresponding 1-form $P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$ is $\mathrm{d}f$. Proposition 2.7 expresses the fact that a 1-form on $\mathbb{R}^3$ is exact if and only if it is closed. It's worth remarking at this stage that Proposition 2.7 need not hold true on a region other than $\mathbb{R}^3$, as the following well-known example from calculus suggests.

Example 2.7. Suppose $U = \mathbb{R}^3 \setminus \{z\text{-axis}\}$, and $\mathbf{F}(x, y, z)$ is the vector field

$$\mathbf{F} = \begin{bmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \\ 0 \end{bmatrix}$$

on U. Then $\operatorname{curl} \mathbf{F} = \mathbf{0}$. Indeed,

$$\begin{aligned}
\operatorname{curl} \mathbf{F} &= \begin{bmatrix} \frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right) \\ \frac{\partial}{\partial z}\left(\frac{-y}{x^2+y^2}\right) - \frac{\partial}{\partial x}(0) \\ \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ \frac{(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2} - \frac{-(x^2+y^2)+y\cdot 2y}{(x^2+y^2)^2} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}.
\end{aligned}$$

But $\mathbf{F}$ is not the gradient of a C^∞ function on U. Recall the theorem from vector calculus that the line integral of the gradient of a function along a curve gives the total change in the value of the function from the start to the end of the curve. In other words, if $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is a curve and $f : \mathbb{R}^3 \to \mathbb{R}$ is a scalar function, then

$$\int_a^b (\nabla f) \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \tag{2.33}$$

Then if $\mathbf{F}$ is the gradient of a smooth scalar function, then the line integral

$$\oint_C \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$$

over any closed curve would become 0. Let us take the closed curve to be the unit circle: $x = \cos t$,

$y = \sin t$, $z = 0$ for $t \in [0, 2\pi]$. Then

$$\begin{aligned}
&\oint_C \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y \\
&= \int_0^{2\pi} -\sin t\,\mathrm{d}\,(\cos t) + \int_0^{2\pi} \cos t\,\mathrm{d}\,(\sin t) \\
&= \int_0^{2\pi} \sin^2 t\,\mathrm{d}t + \int_0^{2\pi} \cos^2 t\,\mathrm{d}t \\
&= 2\pi.
\end{aligned}$$

Hence, although $\operatorname{curl}\mathbf{F} = \mathbf{0}$, there is no C^∞ function f on U such that $\mathbf{F} = \operatorname{grad} f$. In the language of differential forms, the 1-form

$$\omega = \frac{-y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y$$

is closed but not exact.

It turns out that whether Proposition 2.7 is true for a region $U \subseteq \mathbb{R}^3$ depends on the topology of U. One measure of the failure of a closed k-form to be exact is the quotient vector space

$$H^k(U) = \frac{\{\text{closed } k\text{-forms on } U\}}{\{\text{exact } k\text{-forms on } U\}},$$

called the k-th de Rham cohomology of U. The generalization of Proposition 2.7 to any differential form on $\mathbb{R}^n$ is called the **Poincaré lemma**:

> For $k \geq 1$, every closed k-form on $\mathbb{R}^n$ is exact.

This statement is equivalent to the vanishing of the k-th de Rham cohomology $H^k(\mathbb{R}^n)$ for $k \geq 1$.

3 Differential Forms on Manifold

§3.1 Definition and Local Expression

Let M be a smooth manifold and $p \in M$. The **cotangent space** of M at p, denoted by T_p^*M is the dual space of the tangent space T_pM. An element in T_p^*M is called a covector at p. Thus, a covector $\omega_p \in T_p^*M$ is a linear function

$$\omega_p : T_pM \to \mathbb{R}.$$

A 1-form on M is a functiion that assigns to each $p \in M$, a covector at p.

Definition 3.1 (Differential of a function)**.** Let $f : M \to \mathbb{R}$ be a C^∞ function on a manifold M. Its **differential** is defined to be the 1-form $\mathrm{d}f$ on M such that for any $p \in M$ and $X_p \in T_pM$,

$$(\mathrm{d}f)_p (X_p) = X_p f. \tag{3.1}$$

Proposition 3.1

If $f : M \to \mathbb{R}$ is a C^∞ function, then for $p \in M$ and $X_p \in T_pM$,

$$f_{*,p} (X_p) = (\mathrm{d}f)_p (X_p) \left.\frac{\partial}{\partial x}\right|_{f(p)}.$$

Proof. Since $f_{*,p} (X_p) \in T_{f(p)}\mathbb{R}$, there is a real number c such that

$$f_{*,p} (X_p) = c \left.\frac{\partial}{\partial x}\right|_{f(p)}. \tag{3.2}$$

(Here the chart chosen on $\mathbb{R}$ is $(\mathbb{R}, \mathbb{1}_\mathbb{R})$ so that x is the coordinate of this chart, i.e. $x = \mathbb{1}_\mathbb{R}$.) To evaluate c, apply both sides of (3.2) to the function $x \in C^\infty(\mathbb{R})$. Then

$$f_{*,p} (X_p) (x) = c \left.\frac{\partial}{\partial x}\right|_{f(p)} (x) = c.$$

Therefore,

$$c = f_{*,p} (X_p) (x) = X_p (x \circ f) = X_p f = (\mathrm{d}f)_p (X_p), \tag{3.3}$$

since $x = \mathbb{1}_R$. Therefore, substituting the value of c into (3.2),

$$f_{*,p} (X_p) = (\mathrm{d}f)_p (X_p) \left.\frac{\partial}{\partial x}\right|_{f(p)}. \tag{3.4}$$

■

Let $(U, \varphi) \equiv (U, x^1, x^2, \ldots, x^n)$ be a coordinate chart on M. Here $x^i = r^i \circ \varphi$, where r^i is the i-th coordinate function of a vector in $\mathbb{R}^n$. Then the differentials $\mathrm{d}x^1, \mathrm{d}x^2, \ldots, \mathrm{d}x^n$ are 1-forms on U.

Proposition 3.2

At each point $p \in U$, the covectors $(\mathrm{d}x^1)_p, \ldots, (\mathrm{d}x^n)_p$ form a basis for the cotangent space T_p^*M, dual to the basis $\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$ for the tangent space T_pM.

Proof. Observe that

$$\left(\mathrm{d}x^i\right)_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right)=\left.\frac{\partial}{\partial x^j}\right|_p\left(x^i\right)=\delta^i{}_j. \tag{3.5}$$

So $\left\{(\mathrm{d}x^1)_p,\ldots,(\mathrm{d}x^n)_p\right\}$ is the dual basis to $\left\{\left.\frac{\partial}{\partial x^1}\right|_p,\ldots,\left.\frac{\partial}{\partial x^n}\right|_p\right\}$. ■

Thus, every 1-form ω on U can be written as a linear combination

$$\omega=\sum_{i=1}^n a_i\ \mathrm{d}x^i,$$

where a_i are functions on U. In particular, if f is a C^∞ function on M, then the 1-form $\mathrm{d}f$, when restricted to U, must be a linear combination

$$\mathrm{d}f=\sum_{i=1}^n a_i\ \mathrm{d}x^i. \tag{3.6}$$

If we evaluate both sides of (3.6) on $\frac{\partial}{\partial x^j}$,

$$(\mathrm{d}f)\left(\frac{\partial}{\partial x^j}\right)=\sum_{i=1}^n a_i\ \mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right)=\sum_{i=1}^n a_i\delta^i{}_j=a_j.$$

Then

$$a_j=(\mathrm{d}f)\left(\frac{\partial}{\partial x^j}\right)=\frac{\partial f}{\partial x^j}.$$

Therefore,

$$\mathrm{d}f=\sum_{i=1}^n\frac{\partial f}{\partial x^i}\ \mathrm{d}x^i. \tag{3.7}$$

§3.2 The Cotangent Bundle

The underlying set of the **cotangent bundle** is the disjoint union of the cotangent spaces at all points of M:

$$T^*M=\bigsqcup_{p\in M}T_p^*M=\bigcup_{p\in M}\{p\}\times T_p^*M. \tag{3.8}$$

Let us give T^*M a topology in the following way: let $(U,x^1,\ldots,x^n)$ be a chart on M and $p\in U$. Then each $\omega_p\in T_p^*M$ can be written uniquely as a linear combination

$$\omega_p=\sum_{i=1}^n c_i\left(\omega_p\right)\left(\mathrm{d}x^i\right)_p,$$

with $c_i\left(\omega_p\right)\in\mathbb{R}$. This gives rise to a bijection

$$\begin{aligned}\widetilde{\varphi}:T^*U&\to\varphi\left(U\right)\times\mathbb{R}^n\\(p,\omega_p)&\mapsto\left(\varphi\left(p\right),c_1\left(\omega_p\right),c_2\left(\omega_p\right),\ldots,c_n\left(\omega_p\right)\right).\end{aligned}$$

We use this bijection $\widetilde{\varphi}$ to transfer the topology of $\varphi\left(U\right)\times\mathbb{R}^n$ to T^*U: a set $A\subseteq T^*U$ is said to be open if and only if $\widetilde{\varphi}\left(A\right)$ is open in $\varphi\left(U\right)\times\mathbb{R}^n$, where $\varphi\left(U\right)\times\mathbb{R}^n$ is given the subspace topology of $\mathbb{R}^{2n}$. Now, let $\mathscr{A}=\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in I}$ be the maximal atlas of M. Now, let

$$\begin{aligned}\mathcal{B}&=\bigcup_{\alpha\in I}\{A\mid A\subseteq T^*U_\alpha\text{ is open in }A\subseteq T^*U_\alpha\}\\&=\{A\mid A\subseteq T^*U_\alpha\text{ is open in }A\subseteq T^*U_\alpha,\ \alpha\in I\}.\end{aligned}$$

It can be shown using the same technique of tangent bundle that $\mathcal{B}$ forms a basis for topology. We give T^*M the topology generated by the basis $\mathcal{B}$. We declare $A \subseteq T^*M$ to be open if and only if there exists a subfamily $\{B_\lambda\}_\lambda \subseteq \mathcal{B}$ such that

$$A = \bigcup_\lambda B_\lambda.$$

Furthermore, T^*M has the structure of a C^∞ manifold. An atlas for T^*M is

$$\{(T^*U_\alpha, \widetilde{\varphi}_\alpha)\}_{\alpha \in I}.$$

If two coordinate open sets U_α and U_β intersect, suppose $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Then for $p \in U_{\alpha\beta}$, each $\omega_p \in T_p^*M$ has two basis expansions:

$$\omega_p = \sum_{i=1}^n a_i \left(\mathrm{d}x^i\right)_p = \sum_{j=1}^n b_i \left(\mathrm{d}y^i\right)_p. \tag{3.9}$$

(Here $(U_\alpha, x^1, \ldots, x^n)$ and $(U_\beta, y^1, \ldots, y^n)$ are charts.) Now applying $\left.\frac{\partial}{\partial y^k}\right|_p$ to both sides of (3.9),

$$b_k = \sum_{i=1}^n a_i \left(\mathrm{d}x^i\right)_p \left(\left.\frac{\partial}{\partial y^k}\right|_p\right) = \sum_{i=1}^n a_i \left.\frac{\partial x^i}{\partial y^k}\right|_p.$$

Therefore, $\widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1} : \varphi_\alpha(U_{\alpha\beta}) \times \mathbb{R}^n \to \varphi_\beta(U_{\alpha\beta}) \times \mathbb{R}^n$ is given by

$$(\varphi_\alpha(p), a_1, \ldots, a_n) \mapsto \left(\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)(\varphi_\alpha(p)), \sum_{i=1}^n a_i \left.\frac{\partial x^i}{\partial y^1}\right|_p, \ldots, \sum_{i=1}^n a_i \left.\frac{\partial x^i}{\partial y^n}\right|_p\right).$$

$\varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth, and each $\frac{\partial x^i}{\partial y^j}$ is smooth. Therefore, the transition map $\widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1}$ is smooth, making T^*M a smooth manifold.

T^*M is, in fact, a **vector bundle** of rank n over M. It has a natural projection $\pi : T^*M \to M$ given by $(p, \omega_p) \mapsto p$. In terms of cotangent bundle, a 1-form on M is simply a section of the cotangent bundle T^*M, i.e. it is a map $\omega : M \to T^*M$ such that $\pi \circ \omega = \mathbb{1}_M$. We say that a 1-form is **smooth** if it is C^∞ as a map $\omega : M \to T^*M$ between two manifolds.

§3.3 Characterization of Smooth 1-forms

By definition, a 1-form on an open set $U \subseteq M$ is C^∞ if it is C^∞ as a section of the cotangent bundle T^*M over U.

Lemma 3.3

Let $(U, \varphi) \equiv (U, x^1, \ldots, x^n)$ be a chart on a manifold M. A 1-form $\omega = \sum a_i \, \mathrm{d}x^i$ on U is smooth if and only if the coefficient functions a_i are all smooth on U.

Proof. This is a special case of *Proposition 9.4.2* of DG1 which states that:

> Let $\pi : E \to M$ be a C^∞ vector bundle and U an open subset of M. Suppose $s_1, \ldots, s_r$ is a C^∞ frame for E over U. Then a section $s = \sum_{j=1}^r c^j s_j$ of E over U is C^∞ if and only if the coefficients c^j are C^∞ functions on U.

Here we take E to be the cotangent bundle T^*M, and $\{s_i\}_{i=1}^r$ the C^∞ frame for E over U to be the coordinate 1-forms $\{(\mathrm{d}x^i)\}_{i=1}^n$. ■

Proposition 3.4

Let ω be a 1-form on a manifold M. Then the following are equivalent:

(i) ω is C^∞.

(ii) For every point $p \in M$, there is a chart $(U, x^1, \ldots, x^n)$ about p such that if $\omega = \sum_{i=1}^n a_i \, \mathrm{d}x^i$ on U, then the functions a_i are C^∞ on U.

(iii) For any chart $(U, x^1, \ldots, x^n)$ on M, if $\omega = \sum_{i=1}^n a_i \, \mathrm{d}x^i$ on U, then the functions a_i are C^∞ on U.

Proof. **(ii)⇒(i):** By Lemma 3.3, for every point $p \in M$, there is a chart $(U, x^1, \ldots, x^n)$ about p such that ω is smooth on U. In particular, the section $\omega : M \to T^*M$ is smooth at p, for every $p \in M$. Therefore, $\omega : M \to T^*M$ is a smooth map between manifolds.

(i)⇒(iii): If $\omega : M \to T^*M$ is a smooth map between manifolds, ω is smooth at every chart of M. Therefore, by Lemma 3.3, if $\omega = \sum_{i=1}^n a_i \, \mathrm{d}x^i$ on a chart $(U, x^1, \ldots, x^n)$, each a_i is smooth on U.

(iii)⇒(ii): Obvious. ∎

Proposition 3.5

A 1-form ω on a manifold M is C^∞ if and only if for every C^∞ vector field X, the function $\omega(X)$ is C^∞ on M.

Proof. **(⇒)** Suppose ω is a C^∞ 1-form and X is a C^∞ vector field. Let $(U, x^1, \ldots, x^n)$ be a chart on M. Then

$$\omega = \sum_{i=1}^n a_i \, \mathrm{d}x^i \quad \text{and} \quad X = \sum_{i=1}^n b^j \frac{\partial}{\partial x^j}, \tag{3.10}$$

for C^∞ functions a_i and b^j on U. Then on U, one has

$$\omega(X) = \left(\sum_{i=1}^n a_i \, \mathrm{d}x^i \right) \left(\sum_{i=1}^n b^j \frac{\partial}{\partial x^j} \right) = \sum_{i=1}^n \sum_{j=1}^n a_i b^j \delta^i{}_j = \sum_{i=1}^n a_i b^i, \tag{3.11}$$

which is a C^∞ function on U. Since U was chosen to be an arbitrary coordinate open set, $\omega(X)$ is a smooth function on all of M.

(⇐) Suppose ω is a 1-form on M such that for every C^∞ vector field X on M, the function $\omega(X)$ is smooth on M. For a given $p \in M$, choose a coordinate neighborhood $(U, \varphi) \equiv (U, x^1, \ldots, x^n)$ about p. Then one has

$$\omega = \sum_{i=1}^n a_i \, \mathrm{d}x^i$$

on U. Now fix an integer $j \in \{1, 2, \ldots, n\}$. We can extend the C^∞ vector field $\frac{\partial}{\partial x^j}$ on U to a C^∞ vector field X on the whole of M that agrees with $\frac{\partial}{\partial x^j}$ in a neighborhood V of p (not necessarily the whole of U, but possibly a smaller neighborhood) contained in U (*Proposition 11.1.4* of DG1). The extended vector field is defined in the following way: let $\sigma : M \to \mathbb{R}$ be a C^∞ bump function which is identically 1 on a neighborhood V of p and which has support contained in U. Now, define the vector field $q \mapsto X_q \in T_qM$, denoted by X, in terms of the bump function σ in the following way:

$$X_q = \begin{cases} \sigma(q) \left. \frac{\partial}{\partial x^j} \right|_q & \text{if } q \in U, \\ \mathbf{0} & \text{if } q \notin U. \end{cases} \tag{3.12}$$

The vector field X is smooth in the whole of M, as proved in *Proposition 11.1.4* of DG1. Now, by the hypothesis, $\omega(X)$ is C^∞ on M. In particular, $\omega(X)$ is smooth on V. Therefore,

$$\omega(X) = \left(\sum_{i=1}^n a_i \, \mathrm{d}x^i \right) \left(\frac{\partial}{\partial x^j} \right) = \sum_{i=1}^n a_i \delta^i{}_j = a_j$$

is smooth on V. We, therefore, see that the coefficient functions a_i's appearing in $\omega = \sum_{i=1}^n a_i \, \mathrm{d}x^i$ are smooth on $V \subseteq U$. It means that for a given point p, we can find a chart $(V, \widetilde{x}^1, \ldots, \widetilde{x}^n)$, where

$$\widetilde{x}^i = r^i \circ \varphi|_V,$$

such that $\omega = \sum_{i=1}^n a_i|_V \, \mathrm{d}\widetilde{x}^i$ on V, with each $a_i|_V$ smooth on V. Therefore, by Proposition 3.4, ω is C^∞. ■

§3.4 Pullback of 1-forms

Recall that the differential of a smooth map $F : N \to M$ at $p \in N$ is a linear map $F_{*,p} : T_pN \to T_{F(p)}M$ defined by

$$\left[F_{*,p}\left(X_p\right)\right](f) = X_p\left(f \circ F\right), \tag{3.13}$$

where $f \in C^\infty_{F(p)}(M)$. Indeed, $f \circ F \in C^\infty_p(N)$. Analogously, the **codifferential** (the dual of a differential) at $F(p) \in M$ is a linear map

$$F^{*,p} : T^*_{F(p)}M \to T^*_pN.$$

One observes that the differential $F_{*,p}$ pushes forward a tangent vector at $p \in N$ while the codifferential $F^{*,p}$ pulls back a covector from $T^*_{F(p)}M$ at $F(p) \in M$ to T^*_pN.

Remark 3.1. Note that a vector field, in general, cannot be pushed forward under a smooth map $F : N \to M$. Suppose $F : N \to M$ is a smooth map of manifolds. Also suppose $F(p) = F(q) = z \in M$ so that F is not injective. Now, the differentials

$$F_{*,p} : T_pN \to T_zM \text{ and } F_{*,q} : T_qN \to T_zM$$

are linear maps. Now, let $X \in \mathfrak{X}(N)$ be a C^∞ vector field on N so that X_p under $F_{*,p}$ is pushed forward to $F_{*,p}\left(X_p\right) \in T_zM$ and X_q is pushed forward to $F_{*,q}\left(X_q\right) \in T_zM$ under $F_{*,q}$. There is no reason for $F_{*,p}\left(X_p\right)$ and $F_{*,q}\left(X_q\right)$ to be the same tangent vector in T_zM. In other words, in general,

$$F_{*,p}\left(X_p\right) \neq F_{*,q}\left(X_q\right),$$

so that $z \mapsto F_{*,p}\left(X_p\right) := Y_z \in T_zM$ and $z \mapsto F_{*,q}\left(X_q\right) := Y'_z \in T_zM$ are distinct vector fields on M, denoted by Y and Y', respectively. Therefore, if there were push forward of vector fields $F_* : \mathfrak{X}(N) \to \mathfrak{X}(M)$ associated with the non-injective smooth map $F : N \to M$, there is an ambiguity regarding which vector field X gets mapped to.

Furthermore, if F is not surjective, there is $z \in M$ such that $z \neq F(p)$ for any $p \in N$. In that case as well, defining the push forward vector field $F_*(X)$ at the point z is impossible. However, when $F : N \to M$ is a diffeomorphism, one can define the push forward of a vector field.

Contrary to the non-existence of push forward of a vector field associated with a generic smooth map $F : N \to M$, one can always talk about pullback of a 1-form ω on M:

$$\left(F^*\omega\right)_p\left(X_p\right) = \omega_{F(p)}\left(F_{*,p}\left(X_p\right)\right). \tag{3.14}$$

Here, $\omega \in \Omega^1(M)$, $X_p \in T_pN$, $p \in N$. Note that $\left(F^*\omega\right)_p$ is simply the image of the covector $\omega_{F(p)} \in T^*_{F(p)}M$ under the codifferential $F^{*,p} : T^*_{F(p)}M \to T^*_pN$. In other words,

$$\left(F^*\omega\right)_p = F^{*,p}\left(\omega_{F(p)}\right). \tag{3.15}$$

4 Differential k-forms

§4.1 Definition and Local Expression

We denoted by $A_k(V)$ the vector space of alternating k-tensors on V. We have also seen that if $\{\widehat{\alpha}^1, \ldots, \widehat{\alpha}^n\}$ is a basis for 1-tensors on V, then a basis element of $A_k(V)$ is

$$\widehat{\alpha}^{i_1} \wedge \widehat{\alpha}^{i_2} \wedge \cdots \wedge \widehat{\alpha}^{i_k},$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. We apply this construction to the tangent space T_pM of a manifold M at a point $p \in M$. The vector space $A_k(T_pM)$, usually denoted by $\Lambda^k\left(T_p^*M\right)$, is the space of all alternating k-tensors on the tangent space T_pM.

> **Definition 4.1** (Differential k-form)**.** A **differential k-form** on a manifold M is a function ω that assigns to each point $p \in M$, a k-covector $\omega_p \in \Lambda^k\left(T_p^*M\right)$. An n-from on a manifold of dimension n is called a **top degree form**.

Example 4.1. On $\mathbb{R}^n$, at each point p, there is a standard basis for the tangent space $T_p\mathbb{R}^n$:

$$\left\{ \left.\frac{\partial}{\partial r^1}\right|_p, \left.\frac{\partial}{\partial r^2}\right|_p, \ldots, \left.\frac{\partial}{\partial r^n}\right|_p \right\}.$$

Let $\left\{(\mathrm{d}r^1)_p, \ldots, (\mathrm{d}r^n)_p\right\}$ be the dual basis of $T_p^*\mathbb{R}^n$.

$$\left(\mathrm{d}r^i\right)_p \left(\left.\frac{\partial}{\partial r^j}\right|_p \right) = \delta^i{}_j.$$

As p varies over $\mathbb{R}^n$, we get differential forms $\mathrm{d}r^1, \ldots, \mathrm{d}r^n$ on $\mathbb{R}^n$. By Proposition 1.15, a basis element of alternating k-tensors $\Lambda^k\left(T_p^*\mathbb{R}^n\right)$ is

$$\left(\mathrm{d}r^{i_1}\right)_p \wedge \left(\mathrm{d}r^{i_2}\right)_p \wedge \cdots \wedge \left(\mathrm{d}r^{i_k}\right)_p,$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. If ω is a k-form on $\mathbb{R}^n$, then at each point $p \in \mathbb{R}^n$, ω_p is the following linear combination:

$$\omega_p = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} a_{i_1 \cdots i_k} \left(\mathrm{d}r^{i_1}\right)_p \wedge \left(\mathrm{d}r^{i_2}\right)_p \wedge \cdots \wedge \left(\mathrm{d}r^{i_k}\right)_p. \tag{4.1}$$

Omitting the point p, we write

$$\omega = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} a_{i_1 \cdots i_k} \, \mathrm{d}r^{i_1} \wedge \mathrm{d}r^{i_2} \wedge \cdots \wedge \mathrm{d}r^{i_k}. \tag{4.2}$$

In the expression above, $a_{i_1 \cdots i_k}$ are functions on $\mathbb{R}^n$. To simplify the notations, we use multi-indices to write (4.2) as

$$\omega = \sum_I a_I \, \mathrm{d}r^I, \tag{4.3}$$

where $\mathrm{d}r^I = \mathrm{d}r^{i_1} \wedge \mathrm{d}r^{i_2} \wedge \cdots \wedge \mathrm{d}r^{i_k}$, and $I = (i_1, i_2, \ldots, i_k)$ is a strictly ascending multi-index.

Suppose $(U, x^1, \dots, x^n)$ is a coordinate chart on a manifold M. We have already defined the 1-forms $\mathrm{d}x^1, \dots, \mathrm{d}x^n$ on U. Since at each point $p \in U$, $\left\{(\mathrm{d}x^1)_p, \dots, (\mathrm{d}x^n)_p\right\}$ is a basis for T_p^*M, by Proposition 1.15, a basis for $\Lambda^k\left(T_p^*\mathbb{R}^n\right)$ is

$$\left(\mathrm{d}x^{i_1}\right)_p \wedge \left(\mathrm{d}x^{i_2}\right)_p \wedge \cdots \wedge \left(\mathrm{d}x^{i_k}\right)_p,$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Thus, locally a k-form on U will be a linear combination

$$\omega = \sum_I a_I \, \mathrm{d}x^I, \tag{4.4}$$

where $\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$, $I = (i_1, i_2, \dots, i_k)$ is a strictly ascending multi-index, and a_I are functions on U.

§4.2 The Bundle Point of View

Let V be a real vector space. Another common notation for the vector space $A_k(V)$ of alternating k-linear functionos on V is $\Lambda^k(V^*)$.

$$\begin{aligned}
\Lambda^0(V^*) &= A_0(V) = \mathbb{R}, \\
\Lambda^1(V^*) &= A_1(V) = V^*, \\
\Lambda^2(V^*) &= A_2(V),
\end{aligned}$$

and so on. Now, $\Lambda^k(T^*M)$ is defined to be the disjoint union of the vector spaces $\Lambda^k\left(T_p^*M\right)$ as p varies over M. So

$$\begin{aligned}
\Lambda^k(T^*M) &= \bigsqcup_{p \in M} \Lambda^k\left(T_p^*M\right) = \bigsqcup_{p \in M} A_k(T_pM) \\
&= \bigcup_{p \in M} \{p\} \times A_k(T_pM),
\end{aligned} \tag{4.5}$$

which is the set of all alternating k-tensors at all points of M. This set is called the k-th **exterior power** of the cotangent bundle T^*M.

If (U, φ) is a coordinate chart on M, then there is a bijection $\overline{\varphi} : \Lambda^k(T^*U) \to \varphi(U) \times \mathbb{R}^{\binom{n}{k}}$ defined as follows: a generic element of $\Lambda^k(T^*U)$ is (p, ω_p), where $\omega_p \in \Lambda^k\left(T_p^*U\right)$. Then ω_p is a unique linear combination

$$\omega_p = \sum_I a_I(p) \, \mathrm{d}x^I,$$

where I runs over the set of strictly ascending multi-indices of length k. There are $\binom{n}{k}$ many such multi-indices. If we fix a labeling of the multi-indices once and for all, then we have a $\binom{n}{k}$-tuple $(a_I)_I$. Then we define

$$\overline{\varphi}(p, \omega_p) = (\varphi(p), (a_I)_I) \in \varphi(U) \times \mathbb{R}^{\binom{n}{k}}.$$

Thus, $\Lambda^k(T^*U)$ is in a bijective correspondence with $\varphi(U) \times \mathbb{R}^{\binom{n}{k}}$. Using this bijective correspondence, one transfers the topology of $\varphi(U) \times \mathbb{R}^{\binom{n}{k}}$ to $\Lambda^k(T^*U)$. By varying the open set U in the charts contained in the maximal atlas of M, one can obtain a basis that generates the topology on the whole of $\Lambda^k(T^*M)$.

$\Lambda^k(T^*M)$ is defined to be the disjoint union of the vector spaces $\Lambda^k\left(T_p^*M\right)$ as p varies over M. So

$$\Lambda^k(T^*M) = \bigsqcup_{p \in M} \Lambda^k\left(T_p^*M\right). \tag{4.6}$$

If (U, φ) is a coordinate chart on M, then there is a bijection $\overline{\varphi} : \Lambda^k(T^*U) \to \varphi(U) \times \mathbb{R}^{\binom{n}{k}}$ defined as follows: a generic element of $\Lambda^k(T^*U)$ is (p, ω_p), where $\omega_p \in \Lambda^k\left(T_p^*U\right)$. Then ω_p is a unique linear combination

$$\omega_p = \sum_I a_I(p)\, \mathrm{d}x^I,$$

where I runs over the set of strictly ascending multi-indices of length k. There are $\binom{n}{k}$ many such multi-indices. If we fix a labeling of the multi-indices once and for all, then we have a $\binom{n}{k}$-tuple $(a_I)_I$. Then we define

$$\overline{\varphi}(p, \omega_p) = \left(\varphi(p), (a_I)_I\right) \in \varphi(U) \times \mathbb{R}^{\binom{n}{k}}.$$

Thus, $\Lambda^k(T^*U)$ is in a bijective correspondence with $\varphi(U) \times \mathbb{R}^{\binom{n}{k}}$. Using this bijective correspondence, one transfers the topology of $\varphi(U) \times \mathbb{R}^{\binom{n}{k}}$ to $\Lambda^k(T^*U)$. By varying the open set U in the charts contained in the maximal atlas of M, one can obtain a basis that generates the topology on the whole of $\Lambda^k(T^*M)$.

First, let us verify that $\Lambda^k(T^*M)$ is second countable. By *Lemma 9.1.3* of DG1, a manifold M has a countable basis consisting of coordinate open sets. Let $\{U_i\}_i$ be a countable basis for M consisting of coordinate open sets. Let φ_i be the coordinate map on U_i. We have shown that $\Lambda^k(T^*U_i)$ is homeomorphic to $\varphi_i(U_i) \times \mathbb{R}^{\binom{n}{k}}$, which is an open subset of $\mathbb{R}^{n+\binom{n}{k}}$. Hence, $\varphi_i(U_i) \times \mathbb{R}^{\binom{n}{k}}$ is secoond countable. Now, homeomorphism preserves second countability, so $\Lambda^k(T^*U_i)$ is also second countable.

For each i, choose a countable basis $\{B_{i,j}\}_j$ for $\Lambda^k(T^*U_i)$. Then $\{B_{i,j}\}_{i,j}$ is also countable. Now we need to show that $\{B_{i,j}\}_{i,j}$ is a basis for $\Lambda^k(T^*M)$. Let $A \subseteq \Lambda^k(T^*M)$ be open and take $(p, \omega_p) \in A$. We need to show the existence of $B_{i,j}$ such that $(p, \omega_p) \in B_{i,j} \subseteq A$.

Since $\{U_i\}$ is a basis for M, $p \in U_i$ for some i. Then

$$(p, \omega_p) \in \{p\} \times \Lambda^k\left(T_p^*U_i\right) \subseteq \bigcup_{p \in U_i} \{p\} \times \Lambda^k\left(T_p^*U_i\right) = \Lambda^k(T^*U_i)\,.$$

Therefore, $(p, \omega_p) \in A \cap \Lambda^k(T^*U_i)$.

We have used the topology on $\Lambda^k(T^*U_\alpha)$, for U_α being a coordinate open set of M, to define the topology on $\Lambda^k(T^*M)$. So $\Lambda^k(T^*U_\alpha)$ is a subspace of $\Lambda^k(T^*M)$. Since A is open in $\Lambda^k(T^*M)$, $\widetilde{A} := A \cap \Lambda^k(T^*U_i)$ is open in $\Lambda^k(T^*U_i)$. Now, $\widetilde{A}$ is open in $\Lambda^k(T^*U_i)$ and $(p, \omega_p) \in \widetilde{A} = A \cap \Lambda^k(T^*U_i)$. Since $\{B_{i,j}\}_j$ is a basis for $\Lambda^k(T^*U_i)$, there exists some $B_{i,j}$ such that

$$(p, \omega_p) \in B_{i,j} \subseteq \widetilde{A} = A \cap \Lambda^k(T^*U_i) \subseteq A \implies (p, \omega_p) \in B_{i,j} \subseteq A\,.$$

Therefore, the countable collection $\{B_{i,j}\}_{i,j}$ is a basis for TM.

Now we shall prove that $\Lambda^k(T^*M)$ is Hausdorff. Let (p, ω_p) and (q, τ_q) be distinct points of TM.

Case 1: $p \neq q$.

Since M is Hausdorff, there exist disjoint open subsets U_1 and V_1 of M that contain p and q, respectively. Furthermore, there exists coordinate open sets U_2 and V_2 around p and q, respectively. Then $U = U_1 \cap U_2$ and $V = V_1 \cap V_2$ are disjoint coordinate open sets that contain p and q, respectively.

$$(p, \omega_p) \in \{p\} \times \Lambda^k\left(T_p^*M\right) = \{p\} \times \Lambda^k\left(T_p^*U\right) \subseteq \Lambda^k(T^*U)\,.$$

Similarly, $(q, \tau_q) \in \Lambda^k(T^*V)$. Since $U \cap V = \varnothing$, $\Lambda^k(T^*U) \cap \Lambda^k(T^*V) = \varnothing$. Therefore, $\Lambda^k(T^*U)$ and $\Lambda^k(T^*V)$ are the disjoint open subsets of $\Lambda^k(T^*M)$ that contain (p, ω_p) and (q, τ_q), respectively.

Case 2: $p = q$.
Let (U,φ) be a coordinate chart containing p. Then (p,ω_p) and (p,τ_p) are distinct points on $\Lambda^k(T^*U)$, which is homeomorphic to $\varphi(U)\times\mathbb{R}^{\binom{n}{k}}$. $\varphi(U)\times\mathbb{R}^{\binom{n}{k}}$ is Hausdorff, hence so is $\Lambda^k(T^*U)$. Therefore, (p,ω_p) and (p,τ_p) can be separated by open subsets of $\Lambda^k(T^*U)$, which are also open subset of $\Lambda^k(T^*M)$.

Therefore, $\Lambda^k(T^*M)$ is Hausdorff.

So we have verified that $\Lambda^k(T^*M)$ is second countable, Hausdorff, and locally Euclidean. Now we just have to exhibit an atlas on $\Lambda^k(T^*M)$. Let $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in I}$ be an atlas for M. We are now going to show that $\left\{\left(\Lambda^k(T^*U_\alpha),\overline{\varphi}_\alpha\right)\right\}_{\alpha\in I}$ is an atlas for $\Lambda^k(T^*M)$. Clearly,

$$\begin{aligned}\bigcup_{\alpha\in I}\Lambda^k(T^*U_\alpha) &= \bigcup_{\alpha\in I}\bigcup_{p\in U_\alpha}\{p\}\times\Lambda^k\left(T_p^*U_\alpha\right)\\ &= \bigcup_{\alpha\in I}\bigcup_{p\in U_\alpha}\{p\}\times\Lambda^k\left(T_p^*M\right)\\ &= \bigcup_{p\in\bigcup_{\alpha\in I}U_\alpha}\{p\}\times\Lambda^k\left(T_p^*M\right)\\ &= \bigcup_{M}\{p\}\times\Lambda^k\left(T_p^*M\right)=\Lambda^k(T^*M).\end{aligned}$$

So $\left\{\left(\Lambda^k(T^*U_\alpha),\overline{\varphi}_\alpha\right)\right\}_{\alpha\in I}$ indeed covers the whole of $\Lambda^k(T^*M)$. Now we need to show that the charts are compatible. Let $(U,\varphi)=(U,x^1,\dots,x^n)$ and $(V,\psi)=(V,y^1,\dots,y^n)$ be two charts on M such that $U\cap V\neq\varnothing$. We need to show that $\left(\Lambda^k(T^*U),\overline{\varphi}\right)$ and $\left(\Lambda^k(T^*V),\overline{\psi}\right)$ are compatible charts on $\Lambda^k(T^*M)$, i.e. the map $\overline{\psi}\circ\overline{\varphi}^{-1}$ is C^∞.

$$\overline{\psi}\circ\overline{\varphi}^{-1}:\overline{\varphi}(U\cap V)=\varphi(U\cap V)\times\mathbb{R}^{\binom{n}{k}}\to\overline{\psi}(U\cap V)=\psi(U\cap V)\times\mathbb{R}^{\binom{n}{k}}.$$

Let's take a point $(\varphi(p),(a_I)_I)\in\varphi(U\cap V)\times\mathbb{R}^{\binom{n}{k}}$, where $p\in U\cap V$ and a_I are real numbers. Then $\overline{\varphi}^{-1}$ takes it to

$$\left(p,\sum_I a_I\left(\mathrm{d}x^I\right)_p\right)=\left(p,\sum_I a_I\left(\mathrm{d}x^{i_1}\right)_p\wedge\left(\mathrm{d}x^{i_2}\right)_p\wedge\cdots\wedge\left(\mathrm{d}x^{i_k}\right)_p\right).$$

Now, we can write the k-covector $\omega_p=\sum_I a_I\left(\mathrm{d}x^I\right)_p$ in the chart $\left(V,y^1,\dots,y^n\right)$ as follows:

$$\omega_p=\sum_I a_I\left(\mathrm{d}x^I\right)_p=\sum_J b_J\left(\mathrm{d}y^J\right)_p=\sum_J b_J\left(\mathrm{d}y^{j_1}\right)_p\wedge\left(\mathrm{d}y^{j_2}\right)_p\wedge\cdots\wedge\left(\mathrm{d}y^{j_k}\right)_p. \tag{4.7}$$

Now let us evaluate both sides of (4.7) to the tangent vectors $\left.\frac{\partial}{\partial y^{l_1}}\right|_p,\dots,\left.\frac{\partial}{\partial y^{l_k}}\right|_p$ to get:

$$\text{RHS}=\sum_J b_J\left(\mathrm{d}y^{j_1}\right)_p\wedge\cdots\wedge\left(\mathrm{d}y^{j_k}\right)_p\left(\left.\frac{\partial}{\partial y^{l_1}}\right|_p,\dots,\left.\frac{\partial}{\partial y^{l_k}}\right|_p\right)=\sum_J b_J\delta^J K=b_K, \tag{4.8}$$

where $K=(l_1,\dots,l_k)$ is a strictly ascending multi-index.

$$\text{LHS}=\sum_I a_I\left(\mathrm{d}x^{i_1}\right)_p\cdots\wedge\left(\mathrm{d}x^{i_k}\right)_p\left(\left.\frac{\partial}{\partial y^{l_1}}\right|_p,\dots,\left.\frac{\partial}{\partial y^{l_k}}\right|_p\right)=\sum_I a_I\det\left[\frac{\partial x^{i_{d_1}}}{\partial y^{l_{d_2}}}\right]_{d_1,d_2=1}^k. \tag{4.9}$$

Therefore,

$$b_K=\sum_I a_I\det\left[\frac{\partial x^{i_{d_1}}}{\partial y^{l_{d_2}}}\right]_{d_1,d_2=1}^k. \tag{4.10}$$

Now, in the action of $\overline{\psi} \circ \overline{\varphi}^{-1}$

$$\left(\varphi\left(p\right), \left(a_I\right)_I\right) \mapsto \left(\psi\left(p\right), \left(b_K\right)_K\right) = \left(\left(\psi \circ \varphi^{-1}\right)\left(\varphi\left(p\right)\right), \left(\sum_I a_I \det\left[\frac{\partial x^{i_{d_1}}}{\partial y^{l_{d_2}}}\right]^k_{d_1, d_2 = 1}\right)_K\right). \tag{4.11}$$

$\psi \circ \varphi^{-1}$ is smooth, the other components are also smooth, since they are just linear combination of smooth maps. Therefore, $\overline{\psi} \circ \overline{\varphi}^{-1}$ is C^∞, i.e. the charts $\left(\Lambda^k\left(T^*U\right), \overline{\varphi}\right)$ and $\left(\Lambda^k\left(T^*V\right), \overline{\psi}\right)$ are compatible. This proves that $\left\{\left(\Lambda^k\left(T^*U_\alpha\right), \overline{\varphi}_\alpha\right)\right\}_{\alpha \in I}$ is an atlas for $\Lambda^k\left(T^*M\right)$. So $\Lambda^k\left(T^*M\right)$ is a smooth manifold.

$\Lambda^k\left(T^*M\right)$ can, in fact, be shown to be a C^∞ vector bundle of rank $\binom{n}{k}$ over M, i.e. $\pi : \Lambda^k\left(T^*U\right) \to M$ is a C^∞ vector bundle of rank $\binom{n}{k}$ over M. Let $\pi : \Lambda^k\left(T^*M\right) \to M$ be the map that takes (p, ω_p) to $p \in M$. Then $\left(\Lambda^k\left(T^*M\right), M, \pi\right)$ is a vector bundle of rank $r = \binom{n}{k}$.

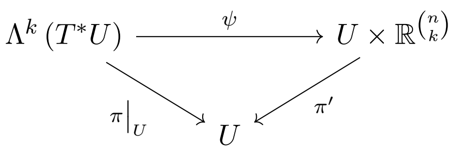
Here, $\pi^{-1}\left(p\right) = \{p\} \times \Lambda^k\left(T_p^*M\right)$, which is a vector space of dimension $\binom{n}{k}$. Each $p \in M$ is contained in a coordinate chart (U, φ), and we have a chart $\left(\Lambda^k\left(T^*U\right), \overline{\varphi}\right)$ on $\Lambda^k\left(T^*M\right)$. So we have a diffeomorphism

$$\overline{\varphi} : \pi^{-1}\left(U\right) = \Lambda^k\left(T^*U\right) \to \varphi\left(U\right) \times \mathbb{R}^{\binom{n}{k}}.$$

$\varphi\left(U\right)$ is diffeomorphic to U, via φ^{-1}. Therefore, we have the following diffeomorphism

$$\psi = \varphi^{-1} \times \mathbb{1}_{\mathbb{R}^{\binom{n}{k}}} \circ \overline{\varphi} : \pi^{-1}\left(U\right) \to U \times \mathbb{R}^{\binom{n}{k}}.$$

This diffeomorphism is fibre-preserving, since the following diagram commutes:



Now, for every $q \in U$,

$$\psi|_{\pi^{-1}(q)} : \pi^{-1}(q) = \{q\} \times \Lambda^k\left(T_q^*U\right) \xrightarrow{\sim} \{q\} \times \mathbb{R}^{\binom{n}{k}}$$

is a vector space isomorphism. Therefore, $\left(\Lambda^k\left(T^*M\right), M, \pi\right)$ is indeed a vector bundle of rank $r = \binom{n}{k}$.

A differential k-form is a section of this vector bundle. We define a k-form to be C^∞ if it is C^∞ as a section of the vector bundle $\Lambda^k\left(T^*M\right)$.

Notation. If $\pi : E \to M$ is a C^∞ vector bundle, then the vector space of C^∞ sections of E is denoted by $\Gamma(E)$, or $\Gamma(M, E)$. The vector space of all C^∞ k-forms, i.e. all C^∞ sections of the bundle $\Lambda^k\left(T^*M\right)$ is usually denoted by $\Omega^k\left(M\right)$. Thus,

$$\Omega^k\left(M\right) = \Gamma\left(\Lambda^k\left(T^*M\right)\right) = \Gamma\left(M, \Lambda^k\left(T^*M\right)\right).$$

Lemma 4.1

Let $\left(U, x^1, \ldots, x^n\right)$ be a chart on a manifold M. A k-form $\omega = \sum a_I dx^I$ on U is smooth if and only if the coefficient functions a_I are all smooth on U.

Proof. A k-form ω is just a section of this vector bundle. Now, given a chart $\left(U, x^1, \ldots, x^n\right)$ on M, the collection $\left\{\mathrm{d}x^I\right\}_I$ of sections (where I runs over the set of strictly ascending multi-indices of length k) is a smooth frame, since the collection $\left\{\left(\mathrm{d}x^I\right)_p\right\}_I$ forms a basis for $\Lambda^k\left(T_p^*M\right)$. Therefore, by *Proposition 9.4.2* of DG1, a section

$$\omega = \sum_I a_I \, \mathrm{d}x^I$$

of $\Lambda^k(T^*M)$ over U is C^∞ if and only if the coefficients a_I are C^∞ functions on U. ■

Proposition 4.2 (Characterization of a smooth k-form)
Let ω be a k-form on a manifold M. The following are equivalent:

(i) The k-form ω is C^∞ on M.

(ii) The manifold M has an atlas such that on every chart $(U,\phi) = (U, x^1, \dots, x^n)$ in the atlas, the coefficients a_I of $\omega = \sum a_I \, \mathrm{d}x^I$ relative to the coordinate frame $\left\{\mathrm{d}x^I\right\}_I$ are all C^∞.

(iii) On every chart $(U,\phi) = (U, x^1, \dots, x^n)$ on M, the coefficients a_I of $\omega = \sum a_I \, \mathrm{d}x^I$ relative to the coordinate frame $\left\{\mathrm{d}x^I\right\}_I$ are all C^∞.

(iv) For any k smooth vector fields $X_1, \dots, X_k$ on M, the function $\omega\left(X_1, \dots, X_k\right)$ is C^∞ on M.

Proof. **(ii)⇒(i):** By Lemma 4.1, for every point $p \in M$, there is a chart $(U, x^1, \dots, x^n)$ about p such that ω is smooth on U. In particular, the section $\omega : M \to \Lambda^k\left(T^*M\right)$ is smooth at p, for every $p \in M$. Therefore, $\omega : M \to T^*M$ is a smooth map between manifolds.

(i)⇒(iii): If $\omega : M \to T^*M$ is a smooth map between manifolds, ω is smooth at every chart of M. Therefore, by Lemma 4.1, if $\omega = \sum_I a_I \, \mathrm{d}x^I$ on a chart $(U, x^1, \dots, x^n)$, each a_i is smooth on U.

(iii)⇒(ii): Obvious.

(iii)⇒(iv): Given a chart $(U,\varphi) = (U, x^1, \dots, x^n)$, $\omega = \sum_I a_I \, \mathrm{d}x^I$, and these coefficient functions a_I are all smooth. Suppose we are given any k smooth vector fields $X_1, X_2, \dots, X_k$ on M. Then on U,

$$X_i = \sum_{j=1}^{n} b_i^j \frac{\partial}{\partial x^j}, \tag{4.12}$$

where each b_i^j are smooth functions on U. Therefore,

$$\begin{aligned}
\omega\left(X_1, \dots, X_k\right) &= \left(\sum_I a_I \, \mathrm{d}x^I\right)\left(X_1, \dots, X_k\right) \\
&= \left(\sum_I a_I \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right)\left(X_1, \dots, X_k\right) \\
&= \sum_I a_I \sum_{\sigma \in S_k} \left(\mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k}\right)\left(X_{\sigma(1)}, \dots, X_\sigma(k)\right).
\end{aligned}$$

Now, using (4.12),

$$\begin{aligned}
\omega\left(X_1, \dots, X_k\right) &= \sum_I a_I \sum_{\sigma \in S_k} \left(\mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k}\right)\left(\sum_{j_1=1}^{n} b_{\sigma(1)}^{j_1} \frac{\partial}{\partial x^{j_1}}, \dots, \sum_{j_k=1}^{n} b_{\sigma(k)}^{j_k} \frac{\partial}{\partial x^{j_k}}\right) \\
&= \sum_I a_I \sum_{\sigma \in S_k} \sum_{j_1, \dots, j_k=1}^{n} b_{\sigma(1)}^{j_1} \cdots b_{\sigma(k)}^{j_k} \left(\mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k}\right)\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_k}}\right) \\
&= \sum_I a_I \sum_{\sigma \in S_k} \sum_{j_1, \dots, j_k=1}^{n} b_{\sigma(1)}^{j_1} \cdots b_{\sigma(k)}^{j_k} \delta^{i_1}{}_{j_1} \cdots \delta^{i_k}{}_{j_k} \\
&= \sum_I \sum_{\sigma \in S_k} a_I b_{\sigma(1)}^{i_1} \cdots b_{\sigma(k)}^{i_k},
\end{aligned}$$

which is a sum of product of smooth functions, hence smooth. Therefore, $\omega\left(X_1, \dots, X_k\right)$ is smooth on U. Since U is an arbitrary coordinate open set of M, $\omega\left(X_1, \dots, X_k\right)$ is smooth on the whole M.

(iv)⇒(ii): Take $p \in M$, and let $(U,\varphi) = (U, x^1, \dots, x^n)$ be a chart about p. For each $j = 1, 2, \dots, n$, we can extend the vector field $\frac{\partial}{\partial x^j}$ to a C^∞ vector field X_j that agrees with $\frac{\partial}{\partial x^j}$ in a neighborhood V of p contained in U (V is not necessarily the whole of U, but possibly a smaller neighborhood).

On V, we can express ω as

$$\omega = \sum_I a_I \,\mathrm{d}\widetilde{x}^I, \tag{4.13}$$

where $\widetilde{x}^i = r^i \circ \varphi\big|_V = x^i\big|_V$. Fix a strictly ascending multi-index $J = (j_1, j_2, \ldots, j_k)$ of length k. Then $\omega(X_{j_1}, \ldots, X_{j_2})$ is smooth on M, by hypothesis. Now, on V,

$$\begin{aligned}\omega(X_{j_1}, \ldots, X_{j_2}) &= \left(\sum_I a_I \,\mathrm{d}\widetilde{x}^I\right)\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right)\\ &= \sum_I a_I \left(\mathrm{d}\widetilde{x}^{i_1} \wedge \cdots \wedge \mathrm{d}\widetilde{x}^{i_k}\right)\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right)\\ &= \sum_I a_I \left(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right)\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right),\end{aligned}$$

since $\widetilde{x}^i$ is nothing but x^i restricted to V. Now,

$$\omega(X_{j_1}, \ldots, X_{j_2}) = \sum_I a_I \left(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right)\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right) = \sum_I a_I \delta^I{}_J = a_J. \tag{4.14}$$

Therefore, a_J is smooth on V. So on the chart $\left(V, \widetilde{x}^1, \ldots, \widetilde{x}^n\right)$, if we write $\omega = \sum_I a_I \,\mathrm{d}\widetilde{x}^I$, the coefficient functions a_I are all smooth. Around each point p, we can find such a chart $\left(V, \widetilde{x}^1, \ldots, \widetilde{x}^n\right)$. ■

Example 4.2. We defined the 0-tensors and the 0-covectors as constants, i.e. for a real vector space V, $A_0(V) = L_0(V) = \mathbb{R}$. Now, recall that

$$\Lambda^k(T^*M) = \bigcup_{p\in M} \{p\} \times \Lambda^k\left(T_p^*M\right).$$

Since $\Lambda^0\left(T_p^*M\right) = \mathbb{R}$ for every $p \in M$, one has

$$\Lambda^0(T^*M) = \bigcup_{p\in M} \{p\} \times \mathbb{R} = M \times \mathbb{R}. \tag{4.15}$$

Hence,

$$\Omega^0(M) = \Gamma\left(\Lambda^0(T^*M)\right) = \Gamma(M, M \times \mathbb{R}). \tag{4.16}$$

A C^∞ section of the 0-th exterior power of the tangent bundle T^*M is nothing but a C^∞ sectio of the globally trivial C^∞ vector bundle $M \times \mathbb{R}$ over M. Such a section maps $p \in M$ to a pair $(p, \sigma(p))$ with $\sigma(p) \in \mathbb{R}$. Therefore, such a section is nothing but a smooth assignment $p \mapsto \sigma(p)$, i.e. $\sigma \in C^\infty(M, \mathbb{R})$. So

$$\Omega^0(M) = \Gamma(M, M \times \mathbb{R}) = C^\infty(M, \mathbb{R}).$$

§4.3 Pullback of k-forms

Let $F : N \to M$ be a smooth map of manifolds. Recall that a 1-form $\omega \in \Omega^1(M)$ can be pulled back to $\Omega^1(N)$ via the pullback $F^* : \Omega^1(M) \to \Omega^1(N)$ defined by

$$\left(F^*\omega\right)_p(X_p) = \omega_{F(p)}\left(F_{*,p}(X_p)\right). \tag{4.17}$$

For 0-forms, i.e. functions, the pullback is defined by composition:

$$N \xrightarrow{F} M \xrightarrow{f} \mathbb{R}$$

Given $f \in C^\infty(M, \mathbb{R})$, its pullback is defined to be

$$F^*(f) = f \circ F \in C^\infty(N, \mathbb{R}), \tag{4.18}$$

so that indeed $F^* : \Omega^0(M) \to \Omega^0(N)$.

For a k-form ω on M, we define its pullback $F^*\omega$ as follows: if $p \in N$ and $X_p^1, X_p^2, \dots, X_p^k \in T_pN$ are k tangent vectors, then

$$(F^*\omega)_p\left(X_p^1, X_p^2, \dots, X_p^k\right) = \omega_{F(p)}\left(F_{*,p}\left(X_p^1\right), F_{*,p}\left(X_p^2\right), \dots, F_{*,p}\left(X_p^k\right)\right). \tag{4.19}$$

Example 4.3. Let $U \subseteq M$ be open, and $\iota : U \to M$ be the inclusion map. For a smooth 0-form on M, i.e. a smooth function $f : M \to \mathbb{R}$, its pullback under ι^* is

$$\iota^* f = f \circ \iota = f\big|_U. \tag{4.20}$$

For a k-form ω on M, its pullback $\iota^*\omega$ is also given by restriction of domain. Indeed, for $p \in U$ and $X_p^1, X_p^2, \dots, X_p^k \in T_pU = T_pM$, $\iota_{*,p}X_p^i = X_p^i$. So

$$\begin{aligned}(\iota^*\omega)_p\left(X_p^1, X_p^2, \dots, X_p^k\right) &= \omega_{\iota(p)}\left(\iota_{*,p}\left(X_p^1\right), \dots, \iota_{*,p}\left(X_p^k\right)\right)\\ &= \omega_p\left(X_p^1, X_p^2, \dots, X_p^k\right).\end{aligned}$$

Therefore,

$$(\iota^*\omega)_p = \omega_p, \tag{4.21}$$

for $p \in U$. As a result, $\iota^*\omega = \omega\big|_U$.

Proposition 4.3 (Linearity of pullback)

Let $F : N \to M$ be a C^∞ map. If ω, τ are k-forms on M and α is a real number, then

(i) $F^*(\omega + \tau) = F^*\omega + F^*\tau$.

(ii) $F^*(\alpha\omega) = \alpha\, F^*\omega$.

Proof. Suppose $F : N \to M$ is C^∞. Then the pullback $F^* : \Omega^*(M) \to \Omega^*(N)$ is defined as follows: if $\omega \in \Omega^k(M)$, $F^*\omega \in \Omega^k(N)$ is defined as:

$$(F^*\omega)_p\left(X_p^1, \dots, X_p^k\right) = \omega_{F(p)}\left(F_{*,p}X_p^1, \dots, F_{*,p}X_p^k\right), \tag{4.22}$$

for $p \in N$, and $X_p^i \in T_pN$.

(a) For $\omega, \tau \in \Omega^k(M)$ and $X_p^1, \dots, X_p^k \in T_pN$,

$$\begin{aligned}(F^*(\omega+\tau))_p\left(X_p^1, \dots, X_p^k\right) &= (\omega+\tau)_{F(p)}\left(F_{*,p}X_p^1, \dots, F_{*,p}X_p^k\right)\\ &= \left(\omega_{F(p)} + \tau_{F(p)}\right)\left(F_{*,p}X_p^1, \dots, F_{*,p}X_p^k\right)\\ &= \omega_{F(p)}\left(F_{*,p}X_p^1, \dots, F_{*,p}X_p^k\right) + \tau_{F(p)}\left(F_{*,p}X_p^1, \dots, F_{*,p}X_p^k\right)\\ &= (F^*\omega)_p\left(X_p^2, \dots, X_p^k\right) + (F^*\tau)_p\left(X_p^2, \dots, X_p^k\right)\end{aligned}$$

Therefore,

$$F^*(\omega+\tau) = F^*\omega + F^*\tau. \tag{4.23}$$

(b) For $\alpha \in \mathbb{R}$ and $X_p^1, \ldots, X_p^k \in T_pN$,

$$\begin{aligned}(F^*(\alpha\omega))_p\left(X_p^1, \ldots, X_p^k\right) &= (\alpha\omega)_{F(p)}\left(F_{*,p}X_p^1, \ldots, F_{*,p}X_p^k\right) \\ &= \alpha\omega_{F(p)}\left(F_{*,p}X_p^1, \ldots, F_{*,p}X_p^k\right) \\ &= \alpha \cdot (F^*\omega)_p\left(X_p^1, \ldots, X_p^k\right).\end{aligned}$$

Therefore,

$$F^*(\alpha\omega) = \alpha F^*\omega. \tag{4.24}$$

■

§4.4 The Wedge Product

If $\omega \in \Omega^k(M)$ and $\tau \in \Omega^l(M)$, then for any $p \in M$, $\omega_p \in \Lambda^k\left(T_p^*M\right)$ and $\tau_p \in \Lambda^l\left(T_p^*M\right)$ and $\omega_p \wedge \tau_p \in \Lambda^{k+l}\left(T_p^*M\right)$. Then we define the wedge product of ω and τ to be the $(k+l)$-form $\omega \wedge \tau$ such that

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p. \tag{4.25}$$

Proposition 4.4

If ω and τ are C^∞ forms on M, then so is $\omega \wedge \tau$.

Proof. Let $(U, x^1, \ldots, x^n)$ be a chart on M. On U,

$$\omega = \sum_I a_I \, \mathrm{d}x^I, \quad \tau = \sum_J b_J \, \mathrm{d}x^J \tag{4.26}$$

for C^∞ functions a_I, b_J on U. Their Wedge product is

$$\begin{aligned}\omega \wedge \tau &= \left(\sum_I a_I \, \mathrm{d}x^I\right) \wedge \left(\sum_J b_J \, \mathrm{d}x^J\right) \\ &= \sum_{I,J} a_I b_J \, \mathrm{d}x^I \wedge \mathrm{d}x^J \ldots \end{aligned} \tag{4.27}$$

In (4.27), $\mathrm{d}x^I \wedge \mathrm{d}x^J = 0$ if I and J have at least an index in common. If I and J are disjoint, i.e., have none of their indices to be common, then

$$\mathrm{d}x^I \wedge \mathrm{d}x^J = \pm \mathrm{d}x^K, \tag{4.28}$$

where $K = I \cup J$ but reordered as an increasing sequence. Thus,

$$\omega \wedge \tau = \sum_K \left(\sum_{I \cup J = K} \pm a_I b_J\right) \mathrm{d}x^K. \tag{4.29}$$

Since the coefficients of $\mathrm{d}x^K$ in (4.29) are C^∞, by Proposition 4.2, $\omega \wedge \tau$ is C^∞ on M. ■

Proposition 4.5 (Pullback of wedge product)

If $F : N \to M$ is a C^∞ map of manifolds and ω are τ are differential forms on M, then

$$F^*(\omega \wedge \tau) = F^*(\omega) \wedge F^*(\tau). \tag{4.30}$$

Proof. If $\omega \in \Omega^k(M), \tau \in \Omega^l(M)$, and $X_p^1, \ldots, X_p^{k+l} \in T_pN$,

$$\begin{aligned}
&(F^*(\omega \wedge T))_p \left(X_p^1, \ldots, X_p^{k+l}\right) \\
&= (\omega \wedge \tau)_{F(p)} \left(F_{*,p}X_p^1, \ldots, F_{*,p}X_p^{k+l}\right) \\
&= \left(\omega_{F(p)} \wedge \tau_{F(p)}\right) \left(F_{*,p}X_p^1, \ldots, F_{*,p}X_p^{k+l}\right) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, \omega_{F(p)} \left(F_{*,p}X_p^{\sigma(1)}, \ldots, F_{*,p}X_p^{\sigma(k)}\right) \tau_{F(p)} \left(F_{*,p}X_p^{\sigma(k+1)}, \ldots, F_{*,p}X_p^{\sigma(k+l)}\right) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+1}} (\operatorname{sgn} \sigma) F^*\omega \left(X_p^{\sigma(1)}, \ldots, X_p^{\sigma(k)}\right) F^*\tau \left(X_p^{\sigma(k+1)}, \ldots, X_p^{\sigma(k+1)}\right) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+1}} (\operatorname{sgn} \sigma) \left(F^*\omega \otimes F^*\tau\right) \left(X_p^{\sigma(1)}, \ldots, X_p^{\sigma(k+l)}\right) \\
&= \frac{1}{k!l!} \mathcal{A} \left(F^*\omega \otimes F^*\tau\right) \left(X_p^1, \ldots, X_p^{k+l}\right) \\
&= \left(F^*(\omega) \wedge F^*(\tau)\right) \left(X_p^1, \ldots, X_p^{k+l}\right).
\end{aligned}$$

Therefore,

$$F^*(\omega \wedge \tau) = F^*(\omega) \wedge F^*(\tau). \tag{4.31}$$

■

We define the vector space $\Omega^*(M)$ of C^∞ differential forms on a manifold M of dimension n to be the direct sum

$$\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M). \tag{4.32}$$

Each element of $\Omega^*(M)$ is uniquely a formal sum $\sum_{i=1}^{r} \omega_{k_i}$ with $\omega_{k_i} \in \Omega^{k_i}(M)$. With the wedge product, the vector space $\Omega^*(M)$ becomes a **graded algebra**, graded by the the degree of differential forms. Proposition 4.3 and Proposition 4.5 tells us that the pullback map $F^* : \Omega^*(M) \to \Omega^*(N)$ is a homomorphism of graded algebras[1].

[1]Note that we haven't yet proved that F^* prerserves smoothness of forms, so we don't yet know that F^* maps $\Omega^k(M)$ into $\Omega^k(N)$. But we shall soon prove this in Theorem 5.6, and once we do that we are all good with the notation.

5 Exterior Derivative

The basic objects in differential geometry are differential forms. Our goal will be to learn how we can differentiate and integrate differential forms on manifolds. Recall that an antiderivation on a graded algebra $A = \bigoplus_{k=0}^{\infty} A^k$ is an $\mathbb{R}$-linear map $D : A \to A$ such that

$$D(\omega \cdot \tau) = (D\omega) \cdot \tau + (-1)^k \omega \cdot (D\tau),$$

for $\omega \in A^k$ and $\tau \in A^l$, and $\cdot$ is the multiplication of the graded algebra. In the graded algebra A, an element of A^k is called a **homogenous element of degree** k. The antiderivation D is of degree m if

$$\deg(D\omega) = \deg \omega + m$$

for all homogenous elements $\omega \in A$.

Now, let M be a manifold and $\Omega^*(M)$ the graded algebra of C^∞ differential forms on M. Now, we'll see that on the graded algebra $\Omega^*(M)$, there is a uniquely and intrinsically defined anti-derivation called exterior derivative.

Definition 5.1 (Exterior derivative). An **exterior derivative** on a manifold M is an $\mathbb{R}$-linear map

$$D : \Omega^*(M) \to \Omega^*(M)$$

such that

(i) D is an antiderivation of degree 1,

(ii) $D \circ D = 0$,

(iii) if f is a C^∞ function and X is a C^∞ vector field on M, then $(Df)(X) = Xf$.

Remark 5.1. Condition (iii) in the definition above says that on 0-forms, i.e. C^∞ functions on M, an exterior derivative agrees with the differential $\mathrm{d}f$ of a function f. We have learned earlier that in a coordinate chart $(U, x^1, \ldots, x^n)$, the 1-form $\mathrm{d}f$ can be expressed as

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i.$$

Hence, in the chart $(U, x^1, \ldots, x^n)$,

$$Df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i.$$

We now prove the existence and uniqueness of the exterior differentiation on a manifold.

Lemma 5.1

Let $D : \Omega^*(M) \to \Omega^*(M)$ be an exterior derivative on M. If $f^1, \ldots, f^k$ are smooth functions on U, then

$$D\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^k\right) = 0.$$

Proof. We prove it by induction on k. The base case $k = 1$ follows trivially from $D \circ D = 0$. Suppose

$D\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}\right) = 0$. Then

$$\begin{aligned}
&D\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^k\right)\\
&= D\left(\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}\right) \wedge Df^k\right)\\
&= D\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}\right) \wedge Df^k + (-1)^{k-1}\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^{k-1}\right) \wedge D\left(Df^k\right)\\
&= 0. \qquad (5.1)
\end{aligned}$$

Therefore, $D\left(Df^1 \wedge Df^2 \wedge \cdots \wedge Df^k\right) = 0$ for any $k \geq 1$. ■

§5.1 Exterior Derivative on a Coordinate Chart

Suppose $(U, x^1, \ldots, x^n)$ is a coordinate chart on a manifold M. Then any k-form ω on U is uniquely a linear combination

$$\omega = \sum_I a_I \, \mathrm{d}x^I,$$

where $a_I \in C^\infty(U)$, and the sum runs over all strictly ascending multi-indices I of length k. The $\mathbb{R}$-linear map $\mathrm{d} : \Omega^*(U) \to \Omega^*(U)$ can be constructed to be an exterior derivative on U. In fact, d acts on a homogenous element $\omega \in \Omega^k(U)$ in the following way:

$$\begin{aligned}
\mathrm{d}\omega = \mathrm{d}\left(\sum_I a_I \, \mathrm{d}x^I\right) &= \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I + (-1)^0 \sum_I a_I \, \mathrm{d}\mathrm{d}x^I\\
&= \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I + \sum_I a_I \, \mathrm{d}\left(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right)\\
&= \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I\\
&= \sum_I \sum_j \frac{\partial a_I}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^I. \qquad (5.2)
\end{aligned}$$

(5.2) suggests that $\mathrm{d}\omega \in \Omega^{k+1}(U)$, and it can be written in the chart $(U, x^1, \ldots, x^n)$ using (5.2). This proves the existence of the exterior derivative $\mathrm{d} : \Omega^*(U) \to \Omega^*(U)$, on an open set U of M. The uniqueness of $\mathrm{d} : \Omega^*(U) \to \Omega^*(U)$ can be shown exactly the same way we proved it for the Euclidean case in Proposition 2.4.

Sometimes we write $\mathrm{d}_U\omega$ instead of $\mathrm{d}\omega$ to emphasize that it is the **unique** exterior derivative on the open set $U \subseteq M$. In other words, if (U, x^i) and (U, y^j) are two charts on M, and $\omega = \sum a_I \, \mathrm{d}x^I = \sum b_J \, \mathrm{d}y^J$, then

$$\mathrm{d}_U\omega = \sum_I \sum_i \frac{\partial a_I}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^I = \sum_J \sum_i \frac{\partial b_J}{\partial y^i} \mathrm{d}y^i \wedge \mathrm{d}y^I. \qquad (5.3)$$

This reveals that the expression $\mathrm{d}_U\omega$ is chart independent.

§5.2 Local Operators

An endomorphism of a vector space W (a linear transformation from W to itself) is often called an operator on W. For example, if $W = C^\infty(\mathbb{R})$, the vector space of C^∞ functions on $\mathbb{R}$, then $\frac{\mathrm{d}}{\mathrm{d}x}$ is an operator on W:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = f'(x).$$

The derivative has the desired property that the value of f' at a point p depends only on the values of f in a small neighborhood of p. More precisely, if $f = g$ on an open set $U \subseteq \mathbb{R}$, then $f' = g'$ on U. We say that the derivative is a local operator on $C^\infty(\mathbb{R})$.

Definition 5.2 (Local operator)**.** An operator $D : \Omega^*(M) \to \Omega^*(M)$ is said to be **local** if for all $k \geq 0$, whenever a k-form $\omega \in \Omega^k(M)$ restricts to 0 on an open set U (i.e. $\omega_p = 0$ at every $p \in U$), then $D\omega \equiv 0$ on U (i.e. $(D\omega)_p = 0$ at every $p \in U$).

An equivalent definition of local operator is that for all $k \geq 0$, whenever two k-forms $\omega, \tau \in \Omega^k(M)$ agree on an open set U, then $D\omega \equiv D\tau$ on U (i.e. $(D\omega)_p = (D\tau)_p$ at every $p \in U$).

Proposition 5.2

Any antiderivation D on $\Omega^*(M)$ is a local operator.

Proof. Suppose $\omega \in \Omega^*(M)$ and $\omega \equiv 0$ on an open subset U. Let $p \in U$. It suffices to show that $(D\omega)_p = 0$. Take a bump function f at p supported in U, i.e. $\operatorname{supp} f \subseteq U$. In particular, $f \equiv 1$ in a neighborhood V of p in U, so that $V \subset \operatorname{supp} f \subseteq U$. Then $f\omega \equiv 0$ on M. This can be seen by noting that if $q \in U$,

$$(f\omega)_q = f(q)\omega_q = 0,$$

since $\omega_q = 0$ by hypothesis. On the other hand, if $q \notin U$, then $q \notin \operatorname{supp} f$, so $f(q) = 0$, which yields

$$(f\omega)_q = f(q)\omega_q = 0.$$

Therefore, $f\omega \equiv 0$ on M. Applying D on $f\omega = f \wedge \omega$, we get

$$D(f\omega) = (Df) \wedge \omega + (-1)^0 f \wedge D\omega. \tag{5.4}$$

By the linearity of D, $D(f\omega) = 0$. Now, we evaluate the RHS of (5.4) at $p \in U$, and use the fact that $f(p) = 1$ and $\omega_p = 0$. As a result,

$$\begin{aligned} & (Df)_p \wedge \omega_p + f(p) \wedge (D\omega)_p = 0 \\ \Longrightarrow \quad & (D\omega)_p = 0. \end{aligned} \tag{5.5}$$

Since $p \in U$ is arbitrary, $D\omega \equiv 0$ on U. ■

Sometimes we are given a differential form τ that is defined only on an open subset U of a manifold M. We can use bump functions to extend τ to a global form $\tilde{\tau}$ on M that agrees with τ near some point.

Proposition 5.3

Suppose τ is a C^∞ differential k-form on an open subset U of M (such a differential form is called a local differential form). For any $p \in U$. There is a C^∞ global form $\tilde{\tau}$ on M (can be defined anywhere on M using its charts) that agrees with τ on a neighborhood of p contained in U.

Proof. Choose a smooth bump function f at p supported in U, i.e. $\operatorname{supp} f \subseteq U$. In particular, $f \equiv 1$ in a neighborhood V of p in U, so that $V \subset \operatorname{supp} f \subseteq U$. Then we define

$$\tilde{\tau}_q = \begin{cases} f(q)\,\tau_q & \text{if } q \in U, \\ \mathbf{0}_{\Lambda^k(T_q^* M)} & \text{if } q \notin U. \end{cases}$$

By the definition of $\tilde{\tau}$, it agrees with τ on V. By *Proposition 9.3.1(ii)* of DG1, $\tilde{\tau}$ is smooth on U. Now, let $q \notin U$. We want to show that $\tilde{\tau}$ is smooth at q.

Since $\operatorname{supp} f \subseteq U$, $q \notin U$ implies $q \in M \setminus U \subseteq M \setminus \operatorname{supp} f$. Since $\operatorname{supp} f$ is closed, $M \setminus \operatorname{supp} f$ is open. Hence, we can find a coordinate chart (W, φ) about q such that $W \subseteq M \setminus \operatorname{supp} f$. Then, for $r \in W$, $\tilde{\tau}_r = \mathbf{0}_{\Lambda^k(T_r^* M)}$. Also, $\left(\Lambda^k(T^*U), \overline{\varphi}\right)$ is a chart on $\Lambda^k(T^*M)$ about $\mathbf{0}_{\Lambda^k(T_r^* M)}$.

$$(\overline{\varphi} \circ \tilde{\tau})(r) = (\varphi(r), \underbrace{0, 0, \ldots, 0}_{\binom{n}{k} \text{ 0-s}}).$$

φ is smooth. Therefore, $\tilde{\tau}$ is smooth on W. In particular, $\tilde{\tau}$ is smooth at q. Since $q \notin U$ was arbitrary, $\tilde{\tau}$ is smooth at every $q \notin U$. Therefore, $\tilde{\tau}$ is smooth on all of M. ■

§5.3 Existence and Uniqueness of an Exterior Differentiation

To define an exterior derivative $\mathrm{d} : \Omega^*(M) \to \Omega^*(M)$, let $\omega \in \Omega^k(M)$ and $p \in M$. Choose a chart $(U, x^1, \ldots, x^n)$ about p. Suppose $\omega = \sum_I a_I \, \mathrm{d}x^I$ on U. Now, $\mathrm{d}\omega$ is supposed to be a $(k+1)$-form on M, i.e. $\mathrm{d}\omega \in \Omega^{k+1}(M)$. Define $\mathrm{d}\omega \in \Omega^{k+1}(M)$ such that at $p \in U$, $(\mathrm{d}\omega)_p$ is expressed as

$$(\mathrm{d}\omega)_p = \left(\sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I \right)_p . \tag{5.6}$$

It needs to be proven that the definition (5.6) is independent of chart. If $(V, y^1, \ldots, y^n)$ is another chart about p, and $\omega = \sum_J b_J \, \mathrm{d}y^J$ on V, then on $U \cap V$,

$$\sum_I a_I \, \mathrm{d}_{U\cap V} x^I = \sum_J b_J \, \mathrm{d}_{U\cap V} y^J,$$

where $\mathrm{d}_{U\cap V}$ is the unique exterior derivative $\mathrm{d}_{U\cap V} : \Omega^*(U \cap V) \to \Omega^*(U \cap V)$. Then by the locality of exterior derivative,

$$\mathrm{d}_{U\cap V} \left(\sum_I a_I \, \mathrm{d}_{U\cap V} x^I \right) = \mathrm{d}_{U\cap V} \left(\sum_J b_J \, \mathrm{d}_{U\cap V} y^J \right). \tag{5.7}$$

Reading off the antiderivation $\mathrm{d}_{U\cap V}$ in the chart $(U \cap V, x^1, \ldots, x^n)$ using (5.6), the LHS of (5.7) can be recast into

$$\sum_I \mathrm{d}_{U\cap V} a_I \, \mathrm{d}_{U\cap V} x^I.$$

On the other hand, the antiderivation $\mathrm{d}_{U\cap V}$ in the chart $(U \cap V, y^1, \ldots, y^n)$ can be expressed using (5.6) to compute the RHS of (5.7):

$$\sum_J \mathrm{d}_{U\cap V} b_J \, \mathrm{d}_{U\cap V} y^J.$$

Therefore,

$$\sum_I \mathrm{d}_{U\cap V} a_I \, \mathrm{d}_{U\cap V} x^I = \sum_J \mathrm{d}_{U\cap V} b_J \, \mathrm{d}_{U\cap V} y^J, \tag{5.8}$$

on $U \cap V$. In particular, for $p \in U \cap V$,

$$\left(\sum_I \mathrm{d}_{U\cap V} a_I \, \mathrm{d}_{U\cap V} x^I \right)_p = \left(\sum_J \mathrm{d}_{U\cap V} b_J \, \mathrm{d}_{U\cap V} y^J \right)_p ,$$

proving that the definition (5.6) is indeed chart independent. As p varies over all of M, (5.6) defines an operator

$$\mathrm{d} : \Omega^*(M) \to \Omega^*(M).$$

It's straightforward to verify that the 3 desired conditions of exterior derivative are fullfilled by the definition (5.6).

Now we prove the uniqueness of exterior derivative. Suppose $D : \Omega^*(M) \to \Omega^*(M)$ is an exterior derivative. We will now show that D coincides with the exterior derivative defined by (5.6).

Let $\omega \in \Omega^k(M)$, and $p \in M$. Choose a chart $(U, x^1, \ldots, x^n)$ about p, and suppose $\omega = \sum_I a_I \, \mathrm{d}x^I$ on U. Extend the functions $a_I, x^1, \ldots, x^n$ to C^∞ functions $\widetilde{a}_I, \widetilde{x}^1, \ldots, \widetilde{x}^n$ that agrees with $a_I, x^1, \ldots, x^n$ in a neighborhood V of p. Define

$$\widetilde{\omega} = \sum_I \widetilde{a}_I \, \mathrm{d}\widetilde{x}^I. \tag{5.9}$$

Then $\omega \equiv \widetilde{\omega}$ on V. Since D is a local operator, one must have $D\omega \equiv D\widetilde{\omega}$ on V. Thus,

$$(D\omega)_p = (D\widetilde{\omega})_p = \left[D \left(\sum_I \widetilde{a}_I \, \mathrm{d}\widetilde{x}^I \right) \right]_p . \tag{5.10}$$

Since D is an exterior derivative operator on $\Omega^* M$, and d is the exterior derivative operator defined by (5.6), for $f \in C^\infty(M)$,

$$(Df)(X) = Xf = (\mathrm{d}f)(X),$$

for any C^∞ vector field X. In particular,

$$D\widetilde{a}_I = \mathrm{d}\widetilde{a}_I, \text{ and } D\widetilde{x}^i = \mathrm{d}\widetilde{x}^i,$$

so that $D\widetilde{x}^I = d\widetilde{x}^I$, for a strictly ascending multi-index I of length k. Hence, (5.10) reduces to

$$\begin{aligned}
(D\omega)_p &= \left[D\left(\sum_I \widetilde{a}_I\,\mathrm{d}\widetilde{x}^I\right)\right]_p \\
&= \left[D\left(\sum_I \widetilde{a}_I\,D\widetilde{x}^I\right)\right]_p \\
&= \left(\sum_I D\widetilde{a}_I \wedge D\widetilde{x}^I\right)_p \\
&= \left(\sum_I \mathrm{d}\widetilde{a}_I \wedge \mathrm{d}\widetilde{x}^I\right)_p .
\end{aligned}$$

Now, since $\widetilde{a}_I = a_I$ and $\widetilde{x}^i = x^i$ in a neighborhood of p, we have $\mathrm{d}\widetilde{a}_I = \mathrm{d}a_I$ and $\mathrm{d}\widetilde{x}^I = \mathrm{d}x^I$ at p. Therefore,

$$(D\omega)_p = \left(\sum_I \mathrm{d}\widetilde{a}_I \wedge \mathrm{d}\widetilde{x}^I\right)_p = \left(\sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I\right)_p = (\mathrm{d}\omega)_p . \tag{5.11}$$

So $D = d$, and hence the exterior derivative is unique.

The restriction of a k-form to a submanifold

Let S be a regular submanifold of a manifold M, and ω is a k-form on M, i.e. $\omega \in \Omega^k(M)$. Then the restriction of ω to S is the k-form $\omega\big|_S$ on S defined by

$$\left(\omega\big|_S\right)_p \left(X_p^1, \dots, X_p^k\right) = \omega_p \left(X_p^1, \dots, X_p^k\right), \tag{5.12}$$

for $X_p^1, \dots, X_p^k \in T_pS \subseteq T_pM$. Thus, $\left(\omega\big|_S\right)_p$ is obtained from ω_p by restricting its domain to $T_pS \times T_pS \times \cdots \times T_pS$ (k-times).

Example 5.1. If S is a smooth curve in $\mathbb{R}^2$ defined by the non-constant function $f(x, y) = 0$ (f could be $x^2 + y^2 - 1$, defining the unit circle in $\mathbb{R}^2$), then

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y$$

is a nonzero 1-form on $\mathbb{R}^2$. But since f is identically 0 on S, $(\mathrm{d}f)\big|_S = 0$. So a nonzero form on M can be restricted to a zero form on a submanifold S.

A form that is not identically zero is called a **nonzero form**. On the other hand, a form ω that is nowhere zero,i.e. $\omega_p \neq 0$ for all $p \in M$, is called a **nowhere vanishing form**.

Example 5.2 (A nowhere vanishing 1-form on S^1)**.** Let S^1 be the unit circle defined by $x^2 + y^2 = 1$ in $\mathbb{R}^2$. The 1-form $\mathrm{d}x$ restricts from $\mathbb{R}^2$ to a 1-form on S^1. When restricted to S^1, the domain of the covector $\left((\mathrm{d}x)\big|_{S^1}\right)_p$ is T_pS^1 instead of $T_p\mathbb{R}^2$:

$$\left((\mathrm{d}x)\big|_{S^1}\right)_p : T_pS^1 \to \mathbb{R}.$$

Now, from $x^2 + y^2 = 1$, one obtains

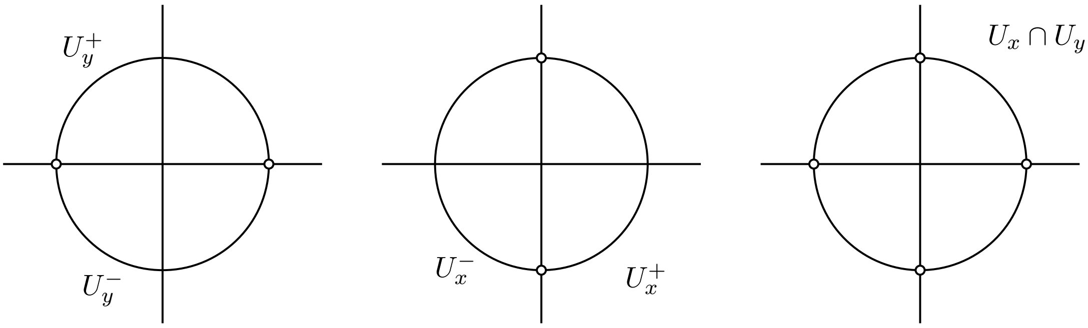
$$2x\,\mathrm{d}x + 2y\,\mathrm{d}y = 0. \tag{5.13}$$

At $p = (1, 0)$, (5.13) reduces to $(\mathrm{d}x)_p = 0$. It shows that although $\mathrm{d}x$ is a nowhere vanishing 1-form on $\mathbb{R}^2$, it vanishes at $(1, 0)$ when restricted to S^1.

To find a nowhere vanishing 1-form on S^1, we again take exterior derivative of both sides of the equation $x^2 + y^2 - 1 = 0$ to arrive at

$$2x\,\mathrm{d}x + 2y\,\mathrm{d}y = 0. \tag{5.14}$$

Let $U_x = \{(x, y) \in S^1 \mid x \neq 0\}$, and $U_y = \{(x, y) \in S^1 \mid y \neq 0\}$.



By (5.14), then one obtains on $U_x \cap U_y$,

$$\frac{\mathrm{d}y}{x} = -\frac{\mathrm{d}x}{y}. \tag{5.15}$$

Now we define a 1-form ω on S^1 by

$$\omega = \begin{cases} \frac{\mathrm{d}y}{x} & \text{on } U_x, \\ -\frac{\mathrm{d}x}{y} & \text{on } U_y. \end{cases} \tag{5.16}$$

Since $\frac{\mathrm{d}y}{x} = -\frac{\mathrm{d}x}{y}$ on $U_x \cap U_y$, ω is a well-defined 1-form on $S^1 = U_x \cup U_y$. To show that ω is C^∞ and nowhere vanishing, we need charts.

$$U_x^+ = \left\{(x, y) \in S^1 \mid x > 0\right\}, U_x^- = \left\{(x, y) \in S^1 \mid x < 0\right\},$$
$$U_y^+ = \left\{(x, y) \in S^1 \mid y > 0\right\}, U_y^- = \left\{(x, y) \in S^1 \mid y < 0\right\}.$$

On U_x^+, the local coordinates are the y-coordinates, so that $(\mathrm{d}y)_p$ is a basis for the cotangent space $T_p^* S^1$ at each $p \in U_x^+$. Now, since $\omega = \frac{\mathrm{d}y}{x}$ on U_x^+, ω is C^∞ and nowhere zero on U_x^+. Similarly, $\omega = \frac{\mathrm{d}y}{x}$ on U_x^- is also C^∞ and nowhere zero on U_x^-. One can show using similar argument that $\omega = -\frac{\mathrm{d}x}{y}$ is C^∞ and nowhere vanishing on U_y^+ and U_y^-. Hence, ω is C^∞ and nowhere zero on S^1.

It's easy to see that this nowhere vanishing smooth 1-form on S^1 is nothing but $x\,\mathrm{d}y - y\,\mathrm{d}x$. On U_x, $x \neq 0$; so using $x\,\mathrm{d}x + y\,\mathrm{d}y = 0$, we get

$$\begin{aligned} x\,\mathrm{d}y - y\,\mathrm{d}x &= x\,\mathrm{d}y - \frac{y}{x}\,x\,\mathrm{d}x = x\,\mathrm{d}y + \frac{y^2}{x}\,\mathrm{d}y \\ &= \left(x + \frac{y^2}{x}\right)\mathrm{d}y = \frac{x^2 + y^2}{x}\,\mathrm{d}y \\ &= \frac{\mathrm{d}y}{x}. \end{aligned} \tag{5.17}$$

On U_y, $y \neq 0$. Again using $x\,\mathrm{d}x + y\,\mathrm{d}y = 0$, we get

$$\begin{aligned} x\,\mathrm{d}y - y\,\mathrm{d}x &= \frac{x}{y}\,y\,\mathrm{d}y - y\,\mathrm{d}x = -\frac{x^2}{y}\,\mathrm{d}x - y\,\mathrm{d}x \\ &= -\left(\frac{x^2}{y} + y\right)\mathrm{d}x = -\frac{x^2+y^2}{y}\,\mathrm{d}x \\ &= -\frac{\mathrm{d}x}{y}. \end{aligned} \tag{5.18}$$

Therefore,

$$x\,\mathrm{d}y - y\,\mathrm{d}x = \omega = \begin{cases} \frac{\mathrm{d}y}{x} & \text{on } U_x, \\ -\frac{\mathrm{d}x}{y} & \text{on } U_y. \end{cases} \tag{5.19}$$

§5.4 Exterior Differentiation Under a Pullback

Theorem 5.4

Let $F : N \to M$ be a smooth map of manifolds. If $\omega \in \Omega^k(M)$, then

$$\mathrm{d}F^*\omega = F^*\mathrm{d}\omega.$$

Proof. Let us first check the case when $k = 0$, i.e. when ω is a 0-form (C^∞ function). We denote this smooth function with h. For $p \in N$ and $X_p \in T_pN$,

$$(\mathrm{d}F^*h)_p(X_p) = X_p(F^*h) = X_p(h \circ F), \tag{5.20}$$

since $(\mathrm{d}f)_p(X_p) = X_p f$ for $f \in C^\infty(M)$. On the other hand,

$$(F^*\mathrm{d}h)_p(X_p) = (\mathrm{d}h)_{F(p)}(F_{*,p}X_p) = (F_{*,p}X_p)(h) = X_p(h \circ F). \tag{5.21}$$

Combining (5.20) and (5.21), we get

$$(\mathrm{d}F^*h)_p = (F^*\mathrm{d}h)_p.$$

Since $p \in N$ is arbitrary,

$$\mathrm{d}F^*h = F^*\mathrm{d}h. \tag{5.22}$$

Now, consider the general case of a C^∞ k-form ω on M, i.e. $\omega \in \Omega^k(M)$. It suffices to verify that $\mathrm{d}F^*\omega = F^*\mathrm{d}\omega$ at an arbitrary point $p \in N$. This reduces the proof to a local computation. If $(V, y^1, \ldots, y^n)$ is a chart of M at $F(p)$, then on V,

$$\omega = \sum_I a_I\,\mathrm{d}y^I = \sum_I a_I\,\mathrm{d}y^{i_1} \wedge \mathrm{d}y^{i_2} \wedge \cdots \wedge \mathrm{d}y^{i_k},$$

for some C^∞ functions a_I on V. Now,

$$F^*\omega = \sum_I (F^*a_I)\left(F^*\mathrm{d}y^{i_1}\right) \wedge \left(F^*\mathrm{d}y^{i_2}\right) \wedge \cdots \wedge \left(F^*\mathrm{d}y^{i_k}\right).$$

Since $\mathrm{d}F^*h = F^*\mathrm{d}h$ for C^∞ function h, we have

$$\begin{aligned} F^*\omega &= \sum_I (a_I \circ F)\,\mathrm{d}\left(F^*y^{i_1}\right) \wedge \mathrm{d}\left(F^*y^{i_2}\right) \wedge \cdots \wedge \mathrm{d}\left(F^*y^{i_k}\right) \\ &= \sum_I (a_I \circ F)\,\mathrm{d}\left(y^{i_1} \circ F\right) \wedge \mathrm{d}\left(y^{i_2} \circ F\right) \wedge \cdots \wedge \mathrm{d}\left(y^{i_k} \circ F\right) \\ &= \sum_I (a_I \circ F)\,\mathrm{d}F^{i_1} \wedge \mathrm{d}F^{i_2} \wedge \cdots \wedge \mathrm{d}F^{i_k}. \end{aligned} \tag{5.23}$$

Therefore, from (5.23), one obtains

$$\mathrm{d}F^*\omega = \sum_I \mathrm{d}\left(a_I \circ F\right) \wedge \mathrm{d}F^{i_1} \wedge \mathrm{d}F^{i_2} \wedge \cdots \wedge \mathrm{d}F^{i_k}. \tag{5.24}$$

On the other hand,

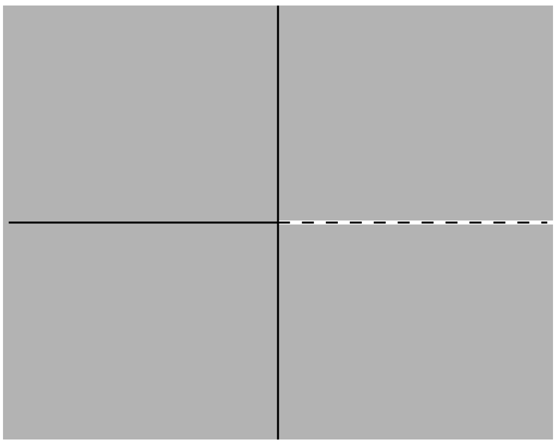
$$\begin{aligned} F^*\mathrm{d}\omega &= F^*\left(\sum_I \mathrm{d}a_I \wedge \mathrm{d}y^{i_1} \wedge \mathrm{d}y^{i_2} \wedge \cdots \wedge \mathrm{d}y^{i_k}\right) \\ &= \sum_I F^*\left(\mathrm{d}a_I\right) \wedge F^*\left(\mathrm{d}y^{i_1}\right) \wedge \cdots \wedge F^*\left(\mathrm{d}y^{i_k}\right) \\ &= \sum_I \mathrm{d}\left(F^*a_I\right) \wedge \mathrm{d}\left(F^*y^{i_1}\right) \wedge \cdots \wedge \mathrm{d}\left(F^*y^{i_k}\right) \\ &= \sum_I \mathrm{d}\left(a_I \circ F\right) \wedge \mathrm{d}\left(y^{i_1} \circ F\right) \wedge \cdots \wedge \mathrm{d}\left(y^{i_k} \circ F\right) \\ &= \sum_I \mathrm{d}\left(a_I \circ F\right) \wedge \mathrm{d}F^{i_1} \wedge \mathrm{d}F^{i_2} \wedge \cdots \wedge \mathrm{d}F^{i_k}. \end{aligned} \tag{5.25}$$

Comparing (5.24) and (5.25), one obtains

$$\mathrm{d}F^*\omega = F^*\mathrm{d}\omega, \tag{5.26}$$

on V. In particular, (5.26) holds at $p \in N$. Since $p \in N$ is arbitrary, (5.26) holds everywhere on N. ■

Example 5.3. Let U be the open set $(0,\infty) \times (0, 2\pi)$ in the (r,θ) plane $\mathbb{R}^2$, i.e. U is $\mathbb{R}^2$ except the non-negative x-axis.



Define $F : U \to \mathbb{R}^2$ by

$$(x, y) = F\left(r, \theta\right) = \left(r\cos\theta, r\sin\theta\right).$$

Let us compute the pullback $F^*\left(\mathrm{d}x \wedge \mathrm{d}y\right)$.

$$F^*\mathrm{d}x = \mathrm{d}F^*x = \mathrm{d}\left(x \circ F\right) = \mathrm{d}\left(r\cos\theta\right) = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta; \tag{5.27}$$

$$F^*\mathrm{d}y = \mathrm{d}F^*y = \mathrm{d}\left(y \circ F\right) = \mathrm{d}\left(r\sin\theta\right) = \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta. \tag{5.28}$$

Therefore,

$$\begin{aligned} F^*\left(\mathrm{d}x \wedge \mathrm{d}y\right) &= F^*\mathrm{d}x \wedge F^*\mathrm{d}y \\ &= \left[\cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta\right] \wedge \left[\sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta\right] \\ &= r\cos^2\theta\,\mathrm{d}r \wedge \mathrm{d}\theta - r\sin^2\theta\,\mathrm{d}\theta \wedge \mathrm{d}r \\ &= r\,\mathrm{d}r \wedge \mathrm{d}\theta. \end{aligned} \tag{5.29}$$

§5.5 Pullback Preserves Smoothness of Forms

In this section, we will prove that if ω is a smooth k-form on M, and $F : N \to M$ is smooth, then $F^*\omega$ is a smooth k-form on N. For that purpose, we need a lemma first.

Lemma 5.5

Let $(U, x^1, \ldots, x^n)$ be a chart on a manifold and $f^1, \ldots, f^k$ smooth functions on U. Then

$$\mathrm{d}f^1 \wedge \cdots \wedge \mathrm{d}f^k = \sum_I \frac{\partial\left(f^1, \ldots, f^k\right)}{\partial\left(x^{i_1}, \ldots, x^{i_k}\right)} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},$$

where $I = (i_1, \ldots, i_k)$ is a strictly ascending multi-index of length k.

Proof. On U,

$$\mathrm{d}f^1 \wedge \cdots \wedge \mathrm{d}f^k = \sum_J c_J \, \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_k}, \tag{5.30}$$

for some functions c_J. By the definition of the differential,

$$\mathrm{d}f^i\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f^i}{\partial x^j}.$$

Applying both sides of (5.30) to the list of coordinate vector fields $\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}$, we get

$$\begin{aligned}\text{LHS} &= \left(\mathrm{d}f^1 \wedge \cdots \wedge \mathrm{d}f^k\right)\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right) = \det\left[\frac{\partial f^i}{\partial x^{i_j}}\right] \\ &= \frac{\partial\left(f^1, \ldots, f^k\right)}{\partial\left(x^{i_1}, \ldots, x^{i_k}\right)},\end{aligned} \tag{5.31}$$

by Proposition 1.13. On the other hand,

$$\text{RHS} = \sum_J c_J \left(\mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_k}\right)\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right) = \sum_J c_J \delta^J_I = c_I. \tag{5.32}$$

Hence, $c_I = \frac{\partial\left(f^1, \ldots, f^k\right)}{\partial\left(x^{i_1}, \ldots, x^{i_k}\right)}$. ■

Theorem 5.6

If $F : N \to M$ is a C^∞ map of manifolds and ω is a C^∞ k-form on M, then $F^*\omega$ is a C^∞ k-form on N.

Proof. It is enough to show that every point in N has a neighborhood on which $F^*\omega$ is C^∞. Fix $p \in N$ and choose a chart $(V, y^1, \ldots, y^m)$ on M about $F(p)$. Let $F^i = y^i \circ F$ be the i-th coordinate of the map F in this chart. By the continuity of F, there is a chart $(U, x^1, \ldots, x^n)$ on N about p such that $F(U) \subset V$. Since ω is C^∞, on V,

$$\omega = \sum_I a_I \, \mathrm{d}y^{i_1} \wedge \cdots \wedge \mathrm{d}y^{i_k}$$

for some C^∞ functions $a_I \in C^\infty(V)$. By properties of the pullback,

$$\begin{aligned} F^*\omega &= F^*\left(\sum_I a_I \,\mathrm{d}y^{i_1} \wedge \cdots \wedge \mathrm{d}y^{i_k}\right) \\ &= \sum_I (F^* a_I)\ F^*\left(\mathrm{d}y^{i_1}\right) \wedge \cdots F^*\left(\mathrm{d}y^{i_k}\right) \\ &= \sum_I (a_I \circ F)\ \mathrm{d}F^* y^{i_1} \wedge \cdots \wedge \mathrm{d}F^* y^{i_k} \\ &= \sum (a_I \circ F)\,\mathrm{d}F^{i_1} \wedge \cdots \wedge \mathrm{d}F^{i_k} \\ &= \sum_{I,J} (a_I \circ F)\,\frac{\partial\left(F^{i_1}, \ldots, F^{i_k}\right)}{\partial\left(x^{j_1}, \ldots, x^{j_k}\right)} \mathrm{d}x^J. \end{aligned} \tag{5.33}$$

Since the $a_I \circ F$ and $\frac{\partial\left(F^{i_1}, \ldots, F^{i_k}\right)}{\partial\left(x^{j_1}, \ldots, x^{j_k}\right)}$ are all C^∞, $F^*\omega$ is C^∞ on U. In particular, $F^*\omega$ is C^∞ at p. Since $p \in N$ is arbitrary, $F^*\omega$ is C^∞ on the whole of N. ■

Theorem 5.7

If $F : N \to M$ and $G : M \to K$ are smooth maps between manifolds, then

$$(G \circ F)^* = F^* \circ G^* : \Omega^*(K) \to \Omega^*(N). \tag{5.34}$$

Furthermore, if $\mathbb{1}_M$ is the identity map on M,

$$(\mathbb{1}_M)^* = \mathbb{1}_{\Omega^*(M)}\,. \tag{5.35}$$

Proof. Suppose $\mathbb{1}_M$ is the identity map on M. Take any $\omega \in \Omega^k(M)$. At any $p \in M$, for any $X_p^1, \ldots, X_p^k \in T_pM$,

$$\begin{aligned} \left((\mathbb{1}_M)^* \omega\right)_p \left(X_p^1, \ldots, X_p^k\right) &= \omega_{\mathbb{1}_M(p)}\left((\mathbb{1}_M)_{*,p} X_p^1, \ldots, (\mathbb{1}_M)_{*,p} X_p^k\right) \\ &= \omega_p\left(\mathbb{1}_{T_pM}\, X_p^1, \ldots, \mathbb{1}_{T_pM}\, X_p^k\right) \\ &= \omega_p\left(X_p^1, \ldots, X_p^k\right), \end{aligned} \tag{5.36}$$

since $(\mathbb{1}_M)_{*,p} = \mathbb{1}_{T_pM}$ by *Remark 6.1.2* of DG1. Therefore, $\left((\mathbb{1}_M)^* \omega\right)_p = \omega_p$. Since $p \in M$ is arbitrary, $(\mathbb{1}_M)^* \omega = \omega$.

Now suppose $F : N \to M$ and $G : M \to K$ are smooth maps between manifolds. Take any $\omega \in \Omega^k(K)$. At any $p \in N$, for any $X_p^1, \ldots, X_p^k \in T_pN$,

$$\begin{aligned} \left((G \circ F)^* \omega\right)_p \left(X_p^1, \ldots, X_p^k\right) &= \omega_{G(F(p))}\left((G \circ F)_{*,p} X_p^1, \ldots, (G \circ F)_{*,p} X_p^k\right) \\ &= \omega_{G(F(p))}\left(G_{*,F(p)}\left(F_{*,p} X_p^1\right), \ldots, G_{*,F(p)}\left(F_{*,p} X_p^k\right)\right), \end{aligned} \tag{5.37}$$

since $(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$ by *Theorem 6.1.1* of DG1. Now on the other hand,

$$\begin{aligned} \left((F^* \circ G^*)\, \omega\right)_p \left(X_p^1, \ldots, X_p^k\right) &= \left(F^*\left(G^*\omega\right)\right)_p \left(X_p^1, \ldots, X_p^k\right) \\ &= (G^*\omega)_{F(p)}\left(F_{*,p} X_p^1, \ldots, F_{*,p} X_p^k\right) \\ &= \omega_{G(F(p))}\left(G_{*,F(p)}\left(F_{*,p} X_p^1\right), \ldots, G_{*,F(p)}\left(F_{*,p} X_p^k\right)\right). \end{aligned} \tag{5.38}$$

Therefore,

$$\left((G \circ F)^* \omega\right)_p \left(X_p^1, \ldots, X_p^k\right) = \left((F^* \circ G^*)\, \omega\right)_p \left(X_p^1, \ldots, X_p^k\right).$$

So we have $\left((G \circ F)^* \omega\right)_p = \left((F^* \circ G^*)\, \omega\right)_p$. Since $p \in N$ is arbitrary,

$$(G \circ F)^* \omega = (F^* \circ G^*)\, \omega. \tag{5.39}$$

■

Remark 5.2. Theorem 5.6 tells us that F^* is indeed a map from $\Omega^k(M)$ to $\Omega^k(N)$. So we can think of it as a map between the graded algebras:

$$F^* : \Omega^* (M) \to F^* (N) .$$

Previously we were writing this without really verifying that F^* preserves the smoothness of forms. Now, by Proposition 4.3 and Proposition 4.5, $F^* : \Omega^*(M) \to \Omega^*(N)$ is a homomorphism of graded algebras. This gives rise to a contravariant functor from the category **Man** of manifolds and smooth maps to the category **GrAlg** of graded algebras and graded algebra homomorphisms:

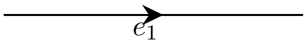
$$\mathcal{F} : \mathbf{Man} \to \mathbf{GrAlg}.$$

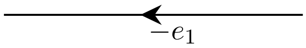
$\mathcal{F}$ takes an object of **Man**, a manifold M, to the graded algebra $\Omega^*(M)$; and it makes an arrow of **Man**, a smooth map $F : N \to M$, to the graded algebra homomorphism $F^* : \Omega^* (M) \to F^* (N)$. Since $\mathcal{F}$ reverses the direction of arrows, Theorem 5.7 ensures that it is a contravariant functor.

6 Orientation

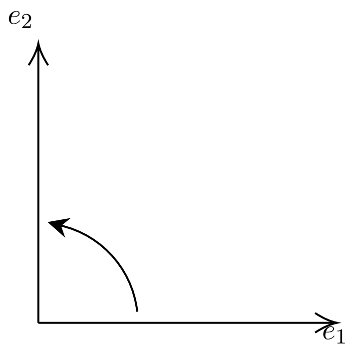
§6.1 Orientations on a Vector Space

On $\mathbb{R}$, an orientation is one of the two possible directions:

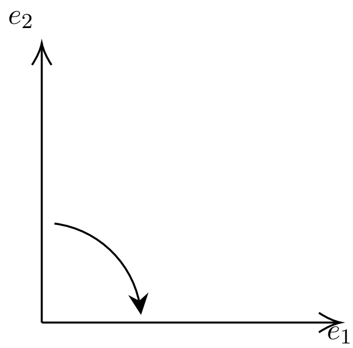




On $\mathbb{R}^2$, an orientation is either counterclockwise or clockwise:

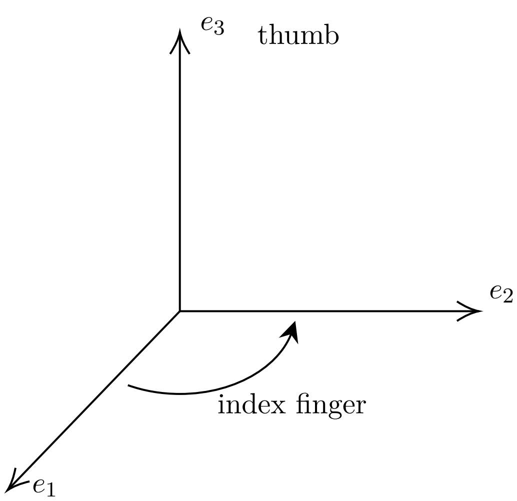


counterclockwise: (e_1, e_2)

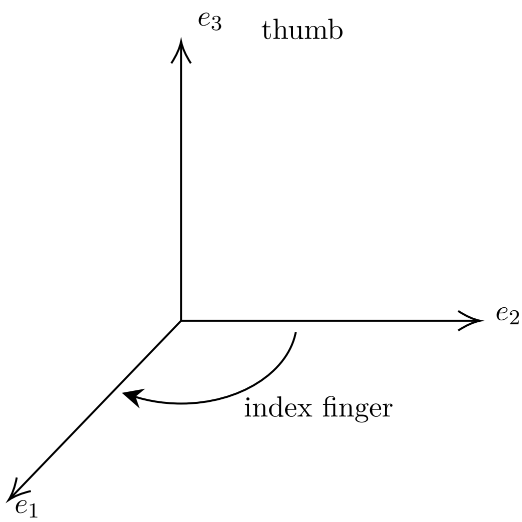


clockwise: (e_2, e_1)

On $\mathbb{R}^3$, an orientation is either right handed or left handed:



Right handed orientation: (e_1, e_2, e_3)



Left handed orientation: (e_2, e_1, e_3)

Now, we want to define an orientation on $\mathbb{R}^4$, or more generally on $\mathbb{R}^n$. We do it through ordered basis for $\mathbb{R}^n$. Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{R}^n$. For $\mathbb{R}^1$, an orientation is given by e_1, or $-e_1$. For $\mathbb{R}^2$, counterclockwise orientation is (e_1, e_2), and clockwise orientation is (e_2, e_1). For $\mathbb{R}^3$, the right handed orientation is (e_1, e_2, e_3), and the left handed orientation is (e_2, e_1, e_3).

For any two ordered bases (u_1, u_2) and (v_1, v_2) of $\mathbb{R}^2$, there is a unique non-singular 2×2 matrix $A = [a_{ij}]$ such that

$$u_j = \sum_{i=1}^{2} v_i \, a_{ij}. \tag{6.1}$$

A is called the change of basis matrix from (v_1, v_2) to (u_1, u_2). In matrix notation, (6.1) can be written as

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} A. \tag{6.2}$$

We say that two ordered bases are **equivalent** if the change of basis matrix A has positive determinant. Then one can check that this is indeed an equivalence relation on the set of all ordered bases of $\mathbb{R}^2$. It, therefore, paritions the ordered bases into two equivalence classes. Each equivalence class is called an **orientation** on $\mathbb{R}^2$.

The equivalence class containing (e_1, e_2) is the counterclockwise orientation, and the equivalence class containing (e_2, e_1) is called clockwise orientation. Indeed,

$$\begin{bmatrix} e_1 & e_2 \end{bmatrix} = \begin{bmatrix} e_2 & e_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and

$$\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1.$$

Similarly, for $\mathbb{R}^3$, the ordered bases (e_1, e_2, e_3) and (e_2, e_1, e_3) don't belong to the same equivalence class:

$$\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} e_2 & e_1 & e_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1.$$

The general case for an n-dimensional vector space V is as follows:

Definition 6.1. Two ordered bases $u = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 & \cdots & u_n \end{bmatrix}$ of an n-dimensional vector space V are said to be **equivalent** if

$$u = vA,$$

for an $n \times n$ matrix A with $\det A > 0$. An **orientation** on V is an equivalence class of ordered bases.

The 0-dimensional vector space $\{\mathbf{0}\}$ is a special case as its basis is the empty set $\varnothing$. We define an orientation on $\{\mathbf{0}\}$ to be one of the two numbers ± 1.

§6.2 Orientations and n-covectors

Instead of using an ordered basis, we can also use an n-covector to specify an orientation on an n-dimensional vector space V. This is based onthe fact that the vector space $\Lambda^n(V^*)$ of n-covectors on V is 1-dimensional (so that it has 2 orientations).

Lemma 6.1

Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be vectors in a vector space V. Suppose

$$u_j = \sum_{i=1}^{n} v_i a_{ij},$$

for a matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ of real numbers. If ω is an n-covector on V, then

$$\omega(u_1, \ldots, u_n) = \det A \omega(v_1, \ldots, v_n).$$

Proof. By hypothesis, $u_j = \sum_{i=1}^{n} v_i a_{ij}$. Using linearity of ω, one arrives at

$$\begin{aligned}\omega\left(u_1, u_2, \ldots, u_n\right) &= \omega\left(\sum_{i_1=1}^{n} v_{i_1} a_{i_1 1}, \sum_{i_2=1}^{n} v_{i_2} a_{i_2 1}, \ldots, \sum_{i_n=1}^{n} v_{i_n} a_{i_n n}\right) \\ &= \sum_{i_1, i_2, \ldots, i_n=1}^{n} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} \omega\left(v_{i_1}, v_{i_2}, \ldots, v_{i_n}\right). \end{aligned} \tag{6.3}$$

For $\omega\left(v_{i_1}, v_{i_2}, \ldots, v_{i_n}\right)$ to be nonzero, $i_1, \ldots, i_n$ must all be different as ω is alternating. Therefore, $i_1, \ldots, i_n$ can be thought of as a permutation σ of $\{1, 2, \ldots, n\}$ that takes each j to i_j. From the alternating property of ω, one has

$$\omega\left(v_{\sigma(1)}, \ldots, v_{\sigma(n)}\right) = (\operatorname{sgn} \sigma)\, \omega\left(v_1, \ldots, v_n\right). \tag{6.4}$$

Therefore, using (6.3),

$$\begin{aligned}\omega\left(u_1, u_2, \ldots, u_n\right) &= \sum_{\sigma \in S_n} a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n} (\operatorname{sgn} \sigma)\, \omega\left(v_1, \ldots, v_n\right) \\ &= (\det A)\, \omega\left(v_1, \ldots, v_n\right). \end{aligned} \tag{6.5}$$

■

Corollary 6.2

If $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are ordered bases of a vector space V, then

$$\begin{aligned} & \omega\left(u_1, u_2, \ldots, u_n\right) \text{ and } \omega\left(v_1, \ldots, v_n\right) \text{ have the same sign} \\ \iff & \det A > 0 \\ \iff & u_1, \ldots, u_n \text{ and } v_1, \ldots, v_n \text{ are equivalent ordered bases.} \end{aligned}$$

We say that the n-covector represents the orientation $\left(v_1, \ldots, v_n\right)$ if $\omega\left(v_1, \ldots, v_n\right) > 0$. By Corollary 6.2, this notion is well-defined, i.e. independent of the choice of ordered basis $v_1, \ldots, v_n$ from the same equivalence class.

Remark 6.1. $\Lambda^n\left(V^*\right) \cong \mathbb{R}$, so that the set of nonzero n-covectors can be identified with $\mathbb{R} \setminus \{0\}$, which has 2 connected components. Two nonzero n-covectors ω and ω' on V are in the same component if and only if $\omega = a\omega'$ for some real number $a > 0$. Thus, each connected component of $\Lambda^n\left(V^*\right) \setminus \{0\}$ represents an orientation on V.

Example 6.1. Let e_1, e_2 be the standard basis of $\mathbb{R}^2$, and α^1, α^2 its dual basis. Then the 2-covector $\alpha^1 \wedge \alpha^2$ represents the counterclockwise orientation on $\mathbb{R}^2$, since

$$\left(\alpha^1 \wedge \alpha^2\right)\left(e_1, e_2\right) = 1 > 0.$$

Example 6.2. Let $\left.\frac{\partial}{\partial x}\right|_p, \left.\frac{\partial}{\partial y}\right|_p$ be the standard basis for the tangent space $T_p\mathbb{R}^2$, and $(\mathrm{d}x)_p, (\mathrm{d}y)_p$ be the dual basis, i.e. for the basis of $T_p^*\mathbb{R}^2$. Then $(\mathrm{d}x)_p \wedge (\mathrm{d}y)_p$ represents the counterclockwise orientation on $T_p\mathbb{R}^2$.

We define an equivalence relation on the nonzero n-covectors on the n-dimensional vector space V as follows:

$$\omega \sim \omega' \iff \omega = a\omega' \text{ for some } a > 0.$$

Then an orientation on V is also given by ana equivalence class of nonzero n-covectors on V.

§6.3 Orientations on a Manifold

Every vector space of dimension n has two orientations, corresponding to the two equivalence classes of ordered bases or the two equivalence classes of nonzero n-covectors. To orient a manifold M, we orient the tangent space at each point $p \in M$ in a coherent way so that the orientation doesn't change abruptly in a neighborhood of a point. The simplest way to guarantee this is to require that the n-form (or top degree form) on M specifying the orientation at each point be C^∞. We also want the n-form to be nowhere vanishing.

Definition 6.2. A manifold M of dimension n is **orientable** if it has a C^∞ nowhere vanishing n-form. If ω is a nowhere vanishing C^∞ n-form on M, then at each point $p \in M$, the n-covector ω_p picks out an equivalence class of ordered bases for the tangent space T_pM.

Example 6.3. The Euclidean space $\mathbb{R}^n$ is orientable as a manifold, because it has the nowhere vanishing n-form $\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n$.

If ω and ω' are both C^∞ nowhere vanishing n-forms on a manifold M of dimension n, then $\omega = f\omega'$ for a C^∞ nowhere vanishing function f on M. On a connoted manifold M, such a function f is either everywhere positive or everywhere negative. Thus, the C^∞ nowhere-vanishing n-forms on a connected manifold M can be partitioned into 2 equivalence class:

$$\omega \sim \omega' \iff \omega = f\omega' \text{ with } f > 0. \tag{6.6}$$

We call either equivalence class an orientation on the connected manifold M. Thus, by definition, a connected manifold has exactly 2 orientations. If the manifold M is not connected, then each connected component of M has one of the 2 possible orientations. We call a C^∞ nowhere-vanishing n-form on M that specifies an orientation of M an **orientation form**. An **oriented manifold** is a pair $(M, [\omega])$, where M is a manifold of dimension n and $[\omega]$ is an orientation on M, i.e. the equivalence class of nowhere vanishing C^∞ n-forms containing ω.

Remark 6.2 (Orientations on a 0-dimensional manifold)**.** A zero dimensional manifold is a point, and by definition is always orientable. Its two orientations are represented by the numbers ± 1.

Definition 6.3. A diffeomorphism $F : (N, [\omega_N]) \to (M, [\omega_M])$ of oriented manifolds is said to be **orientation preserving** if $[F^*\omega_M] = [\omega_N]$. It's **orientation reversing** if $[F^*\omega_M] = [-\omega_N]$.

Proposition 6.3

Let U and V be open subsets of $\mathbb{R}^n$, A diffeomorphiam $F : U \to V$ is orientation-preserving if and only if the Jacobian determinant $\det\left[\frac{\partial F^i}{\partial x^j}\right]$ is everywhere positive on U.

Proof. Let $(x^1, \ldots, x^n)$ and $y', \ldots, y^n$ be standard coordinates on $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$, respectively.

$$\begin{aligned} F^*\left(\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n\right) &= F^*\left(\mathrm{d}y^1\right) \wedge \cdots \wedge F^*\left(\mathrm{d}y^n\right) \\ &= \mathrm{d}\left(F^*y^1\right) \wedge \cdots \wedge \wedge \mathrm{d}^*\left(F^*y^n\right) \\ &= \mathrm{d}\left(y^1 \circ F\right) \wedge \cdots \wedge \mathrm{d}\left(y^n \circ F\right) \\ &= \mathrm{d}F^1 \wedge \cdots \wedge \mathrm{d}F^n \\ &= \det\left[\frac{\partial F^i}{\partial x^j}\right] \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n, \end{aligned} \tag{6.7}$$

where the last equality follows from Lemma 5.5. Now, F is orientation preserving if and only if

$$F^*\left(\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n\right) \sim \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k, \tag{6.8}$$

where $\sim$ is defined as (6.6). Using (6.7), we can conclude that (6.8) holds if and only if $\det\left[\frac{\partial F^i}{\partial x^j}\right]$ is everywhere positive on U. ∎

§6.4 Orientation and Atlases

Definition 6.4 (Oriented atlas)**.** An atlas on M is said to be **oriented** if for any two overlapping charts $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ of the atlas, the Jacobian determinant $\det\left[\frac{\partial y^i}{\partial x^j}\right]$ is everywhere positive on $U \cap V$.

Proposition 6.4

A manifold M of dimension n has a C^∞ nowhere vanishing n-form ω if and only if it has an oriented atlas.

Proof. ($\Leftarrow$) Supoose we are given an oriented atlas

$$\left\{\left(U_\alpha, x_\alpha^1, x_\alpha^2, \ldots, x_\alpha^n\right)\right\}_{\alpha \in A}.$$

Suppose $\{\rho_\alpha\}_{\alpha \in A}$ is a C^∞ partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha \in A}$. Define

$$\omega = \sum \rho_\alpha \, \mathrm{d}x_\alpha^1 \wedge \cdots \wedge \mathrm{d}x_\alpha^n. \tag{6.9}$$

Since $\{\operatorname{supp} \rho_\alpha\}$ is locally finite by deinition of partition of unity, for any $p \in M$, there is an open neighborhood U_p of p that intersects only finitely many of the sets $\operatorname{supp} \rho_\alpha$. Thus, (6.9) is a finite sum on U_p. This actually shows that ω is defined and C^∞ at every point of M.

Let $(U, x^1, \ldots, x^n)$ be one of the charts about p in the oriented atlas. On $U_\alpha \cap U$, by Lemma 5.5,

$$\mathrm{d}x_\alpha^1 \wedge \cdots \wedge \mathrm{d}x_\alpha^n = \det\left[\frac{\partial x_\alpha^i}{\partial x^j}\right] \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \tag{6.10}$$

By hypothesis, $\det\left[\frac{\partial x_\alpha^i}{\partial x^j}\right] > 0$ as the atlas is oriented. Then on $U_p \cap U$,

$$\omega = \sum \rho_\alpha \, \mathrm{d}x_\alpha^1 \wedge \cdots \wedge \mathrm{d}x_\alpha^n = \left(\sum \rho_\alpha \det\left[\frac{\partial x_\alpha^i}{\partial x^j}\right]\right) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \tag{6.11}$$

The sum in (6.11) is a finite sum, since U_p intersects only finitely many of the sets $\operatorname{supp} \rho_\alpha$. Now, it's easy to see that the finite number in the parenthesis is actually positive at p. Indeed, $\det\left[\frac{\partial x_\alpha^i}{\partial x^j}\right] > 0$ at p, since the atlas is oriented. Furthermore, $\rho_\alpha(p) > 0$ for at least one $\alpha \in A$. Hence.

$$\omega_p = (\text{ positive number }) \times \left(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n\right)_p \neq 0.$$

As p is an arbitrary point of M, the n-form ω is nowhere vanishing on M.

($\Rightarrow$) Suppose ω is a C^∞ nowhere vanishing n-form on M. Given an atlas on M, we will use ω to modify the atlas so that it becomes oriented. Without loss of generality, assume that all the open sets of the atlas are connected.

On a chart $(U, x^1, \ldots, x^n)$,

$$\omega = f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \tag{6.12}$$

for a C^∞ function f on U. Since ω is nowhere-vanishing and f is continuous, f is either everywhere positive or everywhere negative on U. If $f > 0$, we leave the chart as it is; if $f < 0$, we replace the chart by $(U, -x^1, x^2, \ldots, x^n)$. After all the charts have been checked and replaced if necessary, we have that on every chart $(V, y^1, \ldots, y^n)$

$$\omega = h \, \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n \tag{6.13}$$

with $h > 0$. This can be seen to be an oriented atlas, since if $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ are two charts, then on $U \cap V$

$$\omega = f\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = h\, \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n, \tag{6.14}$$

with $f, h > 0$. From (6.14),

$$\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n = \frac{f}{h}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \tag{6.15}$$

By Lemma 5.5,

$$\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n = \det\left[\frac{\partial y^i}{\partial x^j}\right] \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \tag{6.16}$$

Comparing (6.15) and (6.16),

$$\det\left[\frac{\partial y^i}{\partial x^j}\right] = \frac{f}{h} > 0 \tag{6.17}$$

on $U \cap V$. Hence, the modified atlas is oriented. ■

Example 6.4 (Non-orientability of the open Möbius band)**.** Let R be the rectangle

$$R = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, \text{ and } -1 < y < 1\right\}.$$

We define an equivalence relation $\sim$ on R as follows:

$$(0, y) \sim (1, -y), \tag{6.18}$$

for $y \in (-1, 1)$. Then $M = R/\!\sim$ is the open Möbius band. We want to show that M is not orientable.

Consider the following open sets on M:

$$\begin{aligned} U &= \{[x, y] \in M \mid 0 < x < 1\}, \\ V &= \left\{[x, y] \in M \mid x \neq \frac{1}{2}\right\}. \end{aligned} \tag{6.19}$$

(Here $[x, y]$ represents the equivalence class containing the point $(x, y) \in R$) Then we can define homeomorphisms $\varphi : U \to \varphi(U) \subset \mathbb{R}^2$ and $\psi : V \to \psi(V) \subset \mathbb{R}^2$:

$$\begin{aligned} \varphi([x, y]) &= (x, y), \\ \psi([x, y]) &= \begin{cases} (x, y) & \text{if } x < \frac{1}{2}, \\ (x - 1, -y) & \text{if } x > \frac{1}{2}. \end{cases} \end{aligned} \tag{6.20}$$

Then $\{(U, \varphi), (V, \psi)\}$ forms an atlas on M. Consider $(U, \varphi) \equiv (U, x^1, x^2)$ and $(V, \psi) = (V, y^1, y^2)$.

Assume for the sake of contradiction that M is orientable. Then there is a nowhere vanishing 2-form ω on M. Then on U,

$$\omega = f\, \mathrm{d}x^1 \wedge \mathrm{d}x^2, \tag{6.21}$$

for a C^∞ nowhere vanishing function f on U. Since U is connected, f is either positive, or negative. Similarly, on V,

$$\omega = g\, \mathrm{d}y^1 \wedge \mathrm{d}y^2, \tag{6.22}$$

for a C^∞ nowhere vanishing function g on V. Since V is connected, g is either positive, or negative. On $U \cap V$,

$$\omega = g\, \mathrm{d}y^1 \wedge \mathrm{d}y^2 = g \det \left[\frac{\partial y^i}{\partial x^i}\right] \mathrm{d}x^1 \wedge \mathrm{d}x^2, \tag{6.23}$$

using Lemma 5.5. Comparing (6.21) and (6.23), we get

$$f = g \det \left[\frac{\partial y^i}{\partial x^i}\right] \tag{6.24}$$

on $U \cap V$. Since f and g are either positive everywhere on $U \cap V$ or negative everywhere on $U \cap V$,

$$\det \left[\frac{\partial y^i}{\partial x^i}\right] = \frac{f}{g} \tag{6.25}$$

is also either positive everywhere on $U \cap V$ or negative everywhere on $U \cap V$. Let us now compute $\det \left[\frac{\partial y^i}{\partial x^i}\right]$.

$$\frac{\partial y^i}{\partial x^j} = \frac{\partial \left(r^i \circ \psi\right)}{\partial x^j} = \frac{\partial \left(\left(r^i \circ \psi\right) \circ \varphi^{-1}\right)}{\partial r^j} = \frac{\partial \left(\psi \circ \varphi^{-1}\right)^i}{\partial r^j}, \tag{6.26}$$

where r^j are coordinates of $\mathbb{R}^2$. Let A, B, C be the following open rectangles in $\mathbb{R}^2$:

$$A = \left(0, \frac{1}{2}\right) \times (-1, 1)\,, \; B = \left(\frac{1}{2}, 1\right) \times (-1, 1)\,, \; C = \left(-\frac{1}{2}, 0\right) \times (-1, 1)\,.$$

Then $\varphi\,(U \cap V) = A \cup B$, $\psi\,(U \cap V) = A \cup C$. $\psi \circ \varphi^{-1} : A \cup B \to A \cup C$ is then

$$\left(\psi \circ \varphi^{-1}\right)\left(r^1, r^2\right) = \begin{cases} (r^1, r^2) & \text{if } (r^1, r^2) \in A, \\ (r^1 - 1, -r^2) & \text{if } (r^1, r^2) \in B. \end{cases} \tag{6.27}$$

So its Jacobian determinant $\det \left[\frac{\partial \left(\psi \circ \varphi^{-1}\right)^i}{\partial r^j}\right]$ is

$$\begin{aligned} \frac{\partial \left(\psi \circ \varphi^{-1}\right)^i}{\partial r^j} &= \begin{cases} \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{on } A, \\ \det \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \text{on } B. \end{cases} \\ &= \begin{cases} 1 & \text{on } A, \\ -1 & \text{on } B. \end{cases} \end{aligned} \tag{6.28}$$

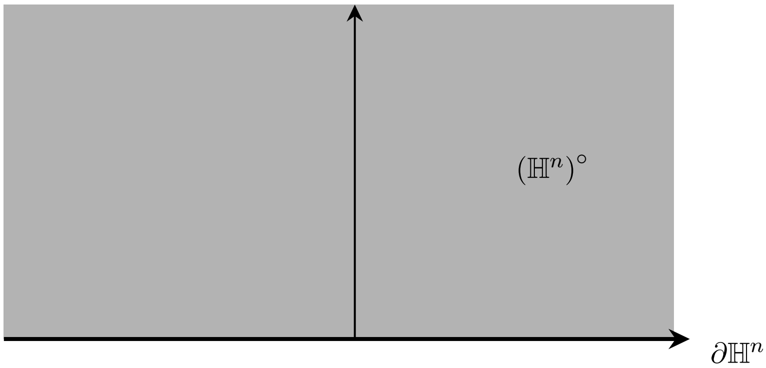
So $\frac{\partial y^i}{\partial x^j}$ is 1 on $\varphi^{-1}(A) \subseteq U \cap V$, and -1 on $\varphi^{-1}(B) \subseteq U \cap V$. But we have previously shown that $\det \left[\frac{\partial y^i}{\partial x^i}\right] = \frac{f}{g}$ is either positive everywhere on $U \cap V$ or negative everywhere on $U \cap V$. Thus we arrive at a contradiction! Hence, no nowhere vanishing 2-form on the open Möbius band M exists.

7 Manifolds with Boundary

The prototype of a manifold with boundary is the closed upper half plane

$$\mathbb{H}^n = \left\{ \left(x^1, \dots, x^n \right) \in \mathbb{R}^n \mid x^n \geq 0 \right\},$$

with the subspace topology inherited from $\mathbb{R}^n$.



$(\mathbb{H}^n)^\circ = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n > 0\}$ is called the interior of $\mathbb{H}^n$; and $\partial\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n = 0\}$ is called the boundary of $\mathbb{H}^n$.

§7.1 Invariance of Domain

Definition 7.1. Let $S \subset \mathbb{R}^n$ be an arbitrary subset (not necessarily open). A function $f : S \to \mathbb{R}^m$ is **smooth** at a point $p \in S$ if there exists a neighborhood U of p in $\mathbb{R}^n$, and a C^∞ function $\widetilde{f} : U \to \mathbb{R}^m$ such that $\widetilde{f} = f$ on $U \cap S$. The function $f : S \to \mathbb{R}^m$ is said to be smooth on S if it is smooth at each point $p \in S$.

Lemma 7.1

A function $f : S \to \mathbb{R}^m$ with $S \subset \mathbb{R}^n$ is C^∞ if and only if there exists an open set $U \subseteq \mathbb{R}^n$ containing S and a C^∞ function $\widetilde{f} : U \to \mathbb{R}^m$ such that $\widetilde{f}\big|_S = f$.

Proof. ($\Leftarrow$) Suppose there is an open set $U \subseteq \mathbb{R}^n$ containing S, and a C^∞ function $\widetilde{f} : U \to \mathbb{R}^m$ such that $\widetilde{f}\big|_S = f$. Then for each $p \in S$, there is a open neighborhood of p, which is U itself, and a C^∞ function $\widetilde{f} : U \to \mathbb{R}^m$ such that $\widetilde{f}$ and f agre on $U \cap S$. In other words, $f : S \to \mathbb{R}^m$ is C^∞ at $p \in S$. Since $p \in S$ was chosen arbitrarily, $f : S \to \mathbb{R}^m$ is C^∞ everywhere on S.

($\Rightarrow$) Suppose $f : S \to \mathbb{R}^m$ is C^∞. Then for each $p \in S$, there is a neighborhood $U_p \subseteq \mathbb{R}^n$ and a C^∞ function $F_p : U_p \to \mathbb{R}^m$ such that $F_p = f$ at $U_p \cap S$. Take

$$U = \bigcup_{p \in S} U_p \subseteq \mathbb{R}^n.$$

Then U is an open subset of $\mathbb{R}^n$ that contains S. Since it is an open subset of an Euclidean space, it is a manifold; and $\{U_p\}_{p \in S}$ is an open cover of U. Therefore, there is a partition of unity $\{\rho_p\}_{p \in S}$ subordinate to the open cover $\{U_p\}_{p \in S}$. Now we define $\widetilde{f} : U \to \mathbb{R}^m$ as

$$\widetilde{f} = \sum_{p \in S} \rho_p F_p. \tag{7.1}$$

Given any $q \in S$, there is a neighborhood V_q of q in U that intersects only finitely many $\operatorname{supp} \rho_p$'s. Therefore, on V_q, the sum in (7.1) becomes a finite sum. Furthermore, as a finite sum and product of smooth functions, $\widetilde{f}$ is smooth on V_q. Therefore, $\widetilde{f} : U \to \mathbb{R}^m$ is smooth.

Now we need to verify that $\widetilde{f}$ agrees with f on S. Let's take any $q \in S$. For $q \in U_p$, since $F_p = f$ at $U_p \cap S$, we have

$$F_p(q) = f(q) . \tag{7.2}$$

And for $q \notin U_p$, $q \notin \operatorname{supp} \rho_p$, since $\operatorname{supp} \rho_p \subseteq U_p$. Therefore,

$$1 = \sum_{p \in S} \rho_p(q) = \sum_{p \in S \text{ such that } q \in \operatorname{supp} \rho_p} \rho_p(q) = \sum_{p \in S \text{ such that } q \in U_p} \rho_p(q) . \tag{7.3}$$

As a result,

$$\begin{aligned} \widetilde{f}(q) &= \sum_{p \in S} \rho_p(q) F_p(q) = \sum_{p \in S \text{ such that } q \in U_p} \rho_p(q) F_p(q) \\ &= \sum_{p \in S \text{ such that } q \in U_p} \rho_p(q) f(q) \\ &= \left(\sum_{p \in S \text{ such that } q \in U_p} \rho_p(q) \right) f(q) \\ &= f(q) . \end{aligned} \tag{7.4}$$

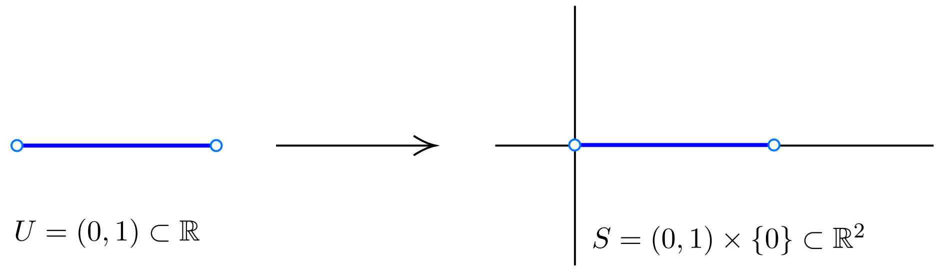
Therefore, $\widetilde{f}\big|_S = f$. ■

Remark 7.1. With the definition above, it now makes sense to speak about an arbitray set $S \subset \mathbb{R}^n$ being diffeomorphic to some set $T \subset \mathbb{R}^m$. This will be the case if and only if there are smooth maps (in the sense above) $f : S \to T \subset \mathbb{R}^m$ and $g : T \to S \subset \mathbb{R}^n$ that are inverses to each other.

Theorem 7.2

Let $U \subseteq \mathbb{R}^n$ be an open subset, $S \subset \mathbb{R}^n$ an arbitrary subset, and $f : U \to S$ a diffeomorphism. Then S is open in $\mathbb{R}^n$.

The diffeomorphism between U ans S forces S to be open in $\mathbb{R}^n$. Given that $f : U \to S$ is a diffeomorphism, we only know that an open subset of U is mapped to an open subset of S under f. Since U is open in itself, $f(U) = S$ is also open in S. We can't immediately conclude that $f(U) = S$ is open in $\mathbb{R}^n$. Besides, it's crucial that both U and S are subsets of the same Euclidean space $\mathbb{R}^n$. For example, there is a diffeomorphism between the open interval $(0, 1) \subset \mathbb{R}$ and the open segment $S = (0, 1) \times \{0\}$ in $\mathbb{R}^2$. But S is not open in $\mathbb{R}^2$.



Proof of Theorem 7.2. Let $f(p) \in S$ be an arbitrary point in S, with $p \in U$. Note that any point in S can be reached this way as f is onto. Since $f : U \to S$ is a diffeomorphism, $f^{-1} : S \to U$ is smooth with S being an arbitrary subset. By Lemma 7.1, there exist an open set $V \subseteq \mathbb{R}^n$ containing S and a C^∞ function $g : V \to \mathbb{R}^n$ such that $g\big|_S = f^{-1}$.

$$U \xrightarrow{f} V \xrightarrow{g} \mathbb{R}^n$$

g agrees f^{-1} on $f(U) = S$. Therefore, $g \circ f = \mathbb{1}_U : U \to U \subseteq \mathbb{R}^n$. Given $p \in U$, by the chain rule, one has

$$g_{*,f(p)} \circ f_{*,p} = \mathbb{1}_{T_pU} : T_pU \to T_pU, \tag{7.5}$$

the identity map on the tangent space T_pU. So $g_{*,f(p)}$ is the left inverse of $f_{*,p}$. The existence of left inverse implies injectivity, so $f_{*,p} : T_pU \to T_{f(p)}V$ is injective. Since U and V are open subsets of the same Euclidean space, they have the same dimension as manifolds. So T_pU and $T_{f(p)}V$ have the same dimension as vector spaces. Now, $f_{*,p} : T_pU \to T_{f(p)}V$ is an injective linear map between vector spaces of same dimension. So $f_{*,p} : T_pU \to T_{f(p)}V$ is invertible.

Now we recall *inverse function theorem*:

> A C^∞ map $F : N \to M$ between two manifolds of same dimension is locally invertible at $p \in N$ (i.e. p has a neighborhood U on which $F|_U : U \to F(U)$ is a diffeomorphism) if and only if the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is an isomorphism of vector spaces.

We have that $f_{*,p} : T_pU \to T_{f(p)}V$ is an isomorphism of vector spaces. Hence, by *inverse function theorem*, $f : U \to V$ is locally invertible at p. This means that there are open neighborhoods U_p of $p \in U$ and $V_{f(p)}$ of $f(p) \in V$ such that

$$f|_{U_p} : U_p \to V_{f(p)}$$

is a diffeomorphism. Then it follows that

$$f(p) \in V_{f(p)} = f(U_p) \subseteq f(U) = S. \tag{7.6}$$

For every $f(p) \in S$, we can find an neighborhood $V_{f(p)} \ni f(p)$ open in $\mathbb{R}^n$ ($V_{f(p)}$ is open in V, V is open in $\mathbb{R}^n$; hence $V_{f(p)}$ is open in $\mathbb{R}^n$) that is contained in S. Therefore, S is open in $\mathbb{R}^n$. ■

Proposition 7.3

Let U and V be open subsets of the upper half space $\mathbb{H}^n$, and $f : U \to V$ be a diffeomorphism. Then f maps interior points to interior points, and boundary points to boundary points.

Proof. Let $p \in U$ be an interior point. Then there is an open ball B in $\mathbb{R}^n$ containing p, which is contained in U. Restriction of a diffeomorphism to an open subset is still a diffeomorphism. Hence,

$$f|_B : B \to f(B)$$

is a diffeomorphism, with B being open in $\mathbb{R}^n$. By Theorem 7.2, $f(B)$ is open in $\mathbb{R}^n$. Hence,

$$f(p) \in f(B) \subseteq f(U) = V \subseteq \mathbb{H}^n.$$

$f(B)$ is open in $\mathbb{R}^n$, and it is contained in $\mathbb{H}^n$. Therefore, $f(B) \subseteq (\mathbb{H}^n)^\circ$. In other words, $f(p) \in (\mathbb{H}^n)^\circ$, since $f(p) \in f(B)$. So f maps interior points to interior points.

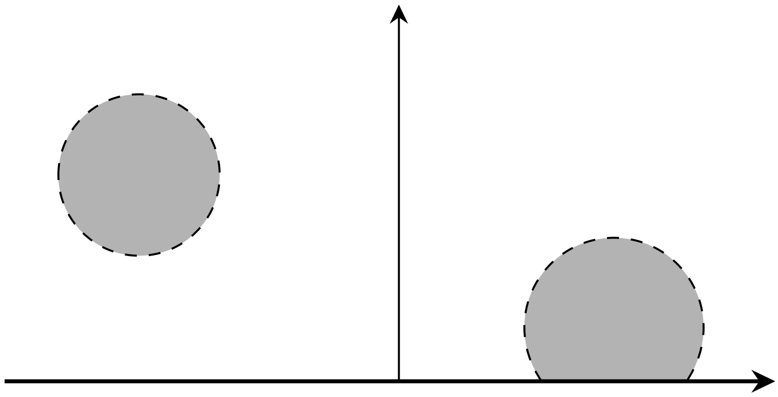
If p is a boundary point in $U \cap \partial\mathbb{H}^n$, then $f^{-1}(f(p)) = p$ is a boundary point. Since $f^{-1} : V \to U$ is a diffeomorphism, by the previous argument, f^{-1} takes interior points to interior points. If $f(p)$ were an interior point, then f^{-1} would have mapped it to an interior point. But f^{-1} maps $f(p)$ to a boundary point. So $f(p)$ cannot be an interior point. Therefore, $f(p)$ is a boundary point. ■

Remark 7.2. Replacing Euclidean spaces by manifolds throughout, one can prove in exactly the same way the **smooth invariance of domain for manifolds**:

> Suppose N and M are n-dimensional manifolds. Let $U \subseteq N$ be open, and $S \subset M$ be any arbitrary subset. If there is a diffeomorphism $F : U \to S$, then S is open in M.

§7.2 Manifolds with Boundary

In the upper half space $\mathbb{H}^n$, there are 2 types of open sets as seen in the following diagram:



In the left one, the set is disjoint from the boundary of $\mathbb{H}^n$, while in the right one the open set has nontrivial intersection with $\partial\mathbb{H}^n$. We say that a topological space M is locally $\mathbb{H}^n$ if every point $p \in M$ has a neighborhood U homeomorphic to an open subset of $\mathbb{H}^n$.

Definition 7.2. A **topological n-manifold with boundary** is a second countable, Hausdorff topological space that is locally $\mathbb{H}^n$.

Let M be a topological n-manifold with boundary. For $n \geq 2$, a chart on M is defined to be a pair (U, φ) consisting of an open set $U \subseteq M$ and a homeomorphism

$$\varphi : U \to \varphi(U) \subseteq \mathbb{H}^n$$

of U with an open subset $\varphi(U)$ of $\mathbb{H}^n$. A slight modification is necessary for the definition of a chart in the case $n = 1$.

Note that $\mathbb{H}^1 = [0, \infty)$ is the right half-line. We also need the left half line $\mathbb{L}^1 = (-\infty, 0]$ to model a 1-manifold with boundary locally. A chart (U, φ) in dimension 1 consists of an open set $U \subseteq M$ and a homeomorphism φ of U with an open subset of $\mathbb{H}^1$ or $\mathbb{L}^1$. With this slight modification of definition of chart in dimension 1, it can be seen that if $\left(U, x^1, x^2, \ldots, x^n\right)$ is a chart of an n-dimensional manifold with boundary, then so is $\left(U, -x^1, x^2, \ldots, x^n\right)$ for $n \geq 1$. A manifold with boundary has dimension at least 1, since a manifold of dimension 0, being a discrete set of points, necessarily has empty boundary.

Definition 7.3. A collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of charts is a C^∞ **atlas** for the topological manifold M with boundary if

$$\bigcup_{\alpha \in A} U_\alpha = M,$$

and for any two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β), the transition map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism. A C^∞ **manifold with boundary** is a topological manifold with boundary together with a maximal C^∞ atlas.

A point $p \in M$ is called an **interior point** in some chart (U, φ) if the point $\varphi(p)$ is an interior point of $\mathbb{H}^n$, i.e. $\varphi(p) \in (\mathbb{H}^n)^\circ$. Similarly, $p \in M$ is a **boundary point** if $\varphi(p)$ is a boundary point of $\mathbb{H}^n$, i.e. $\varphi(p) \in \partial\mathbb{H}^n$. These concepts are independent of charts. Suppose (V, ψ) is another chart about p. Then the diffeomorphism $\psi \circ \varphi^{-1}$ maps $\varphi(p)$ to $\psi(p)$. By Proposition 7.3, $\varphi(p)$ and $\psi(p)$ are both interior points, or both boundary points.

The set of all boundary points of M is denoted by ∂M. On the contrary, the set of all interior points of M is denoted by M°.

In contrast to the geometric notion of the interior and boundary of a manifold, there is the topological notion of the interior and boundary of a subset A of a topological space S. A point $p \in S$ is said to be an interior point of A if there exists an open subset $U \subseteq S$ such that

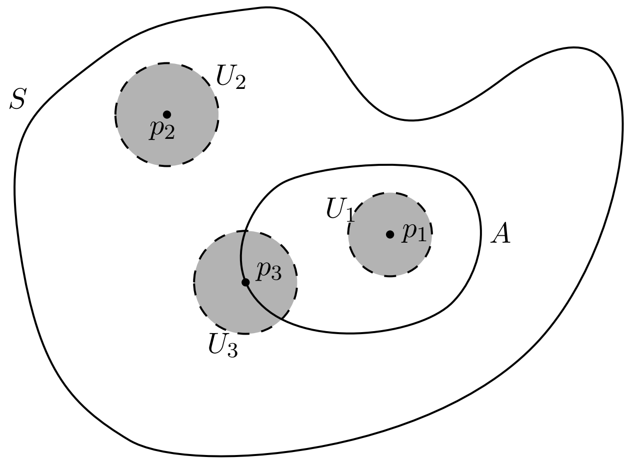
$$p \in U \subseteq A.$$

The point $p \in S$ is an exterior point of A if there exists an open set U of S such that

$$p \in U \subseteq A.$$

Finally $p \in S$ is a boundary point of A if every neighborhood of p contains both a point of A and a point not in A. One denotes by $\operatorname{int}_S(A), \operatorname{ext}_S(A), \operatorname{bd}_S(A)$ the sets of interior, exterior and boundary points of A in S, respectively. Clearly, the topological space S is the disjoint union

$$S = \operatorname{int}_S(A) \sqcup \operatorname{ext}_S(A) \sqcup \operatorname{bd}_S(A). \tag{7.7}$$



In the above diagram, $p_1 \in \operatorname{int}_S(A)$, $p_2 \in \operatorname{ext}_S(A)$, $p_3 \in \operatorname{bd}_S(A)$.

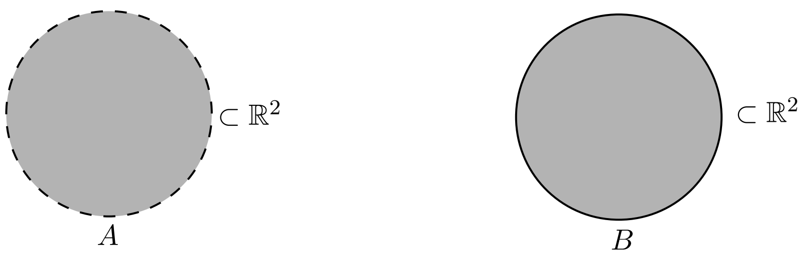
In case the subset $A \subseteq S$ is a manifold with boundary, we call $\operatorname{int}_S(A)$ the topological interior and $\operatorname{bd}_S(A)$ the topological boundary of A, to distinguish them from the manifold interior A° and the manifold boundary ∂A.

Note that the topological interior and the topological boundary of a set depend on an ambient space, while the manifold interior and the manifold boundary are both intrinsic.

Example 7.1 (Topological boundary vs. manifold boundary)**.** Let A be the open unit disk in $\mathbb{R}^2$:

$$A = \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| < 1 \right\}.$$

Then its topological boundary $\operatorname{bd}_{\mathbb{R}^2} A$ in $\mathbb{R}^2$ is the unit circle, while A being a 2-dimensional manifold (without boundary) has its manifold boundary to be the empty set $\varnothing$.



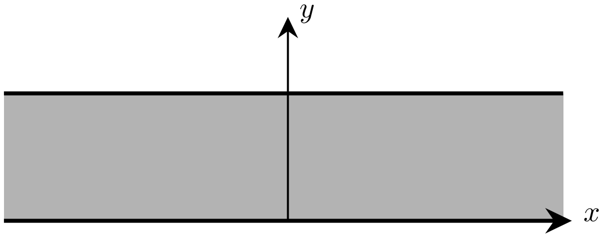
Now, consider B to be the closed unit disk in $\mathbb{R}^2$:

$$B = \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| \leq 1 \right\}.$$

It is a 2-dimensional manifold with boundary, with its manifold boundary ∂B being the unit circle. The topological boundary $\operatorname{bd}_{\mathbb{R}^2}(B)$ of B in $\mathbb{R}^2$ is also the unit circle and hence ∂B and $\operatorname{bd} B$ coincide with each other.

Example 7.2 (Topological interior vs. manifold interior)**.** Let S be the upper half-plane $\mathbb{H}^2$, and A be the subset

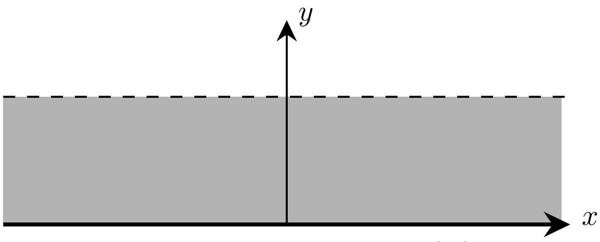
$$A = \left\{(x, y) \in \mathbb{H}^2 \mid y \leq 1\right\}.$$



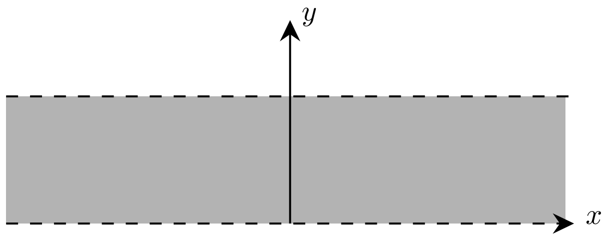
The topological interior $\operatorname{int}_{\mathbb{H}^2}(A)$ of A in $\mathbb{H}^2$ is the set

$$\operatorname{int}_{\mathbb{H}^2}(A) = \left\{(x, y) \in \mathbb{H}^2 \mid 0 \leq y < 1\right\},$$

containing the x-axis.



Topological interior $\operatorname{int}_{\mathbb{H}^2}(A)$



Manifold interior A°

On the other haand, the manifold interior A° of the 2-dimensional manifold with boundary A is the set

$$A^\circ = \left\{(x, y) \in \mathbb{H}^2 \mid 0 < y < 1\right\},$$

not containing the x-axis.

Let us now consider the same set A, but now as a subset of $\mathbb{R}^2$ instead of $\mathbb{H}^2$:

$$A = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\right\}.$$

Then the topological interior $\operatorname{int}_{\mathbb{R}^2}(A)$ of A in $\mathbb{R}^2$ is the set

$$\operatorname{int}_{\mathbb{R}^2}(A) = \left\{(x, y) \in \mathbb{H}^2 \mid 0 < y < 1\right\},$$

which coincides with A°.

The boundary of a manifold with boundary

Let M be a manifold of dimension n with boundary ∂M. If (U, φ) is a chart of M, we denote by

$$\varphi' = \varphi|_{U \cap \partial M},$$

the restriction of the coordinate map φ to the boundary ∂M. Since φ maps boundary points to boundary points,

$$\varphi' = \varphi|_{U \cap \partial M} : U \cap \partial M \to \partial \mathbb{H}^n = \mathbb{R}^{n-1}.$$

Additionally, if (U, φ) and (V, ψ) are two charts on M, then

$$\psi' \circ (\varphi')^{-1} : \varphi(U \cap V \cap \partial M) \to \psi(U \cap V \cap \partial M)$$

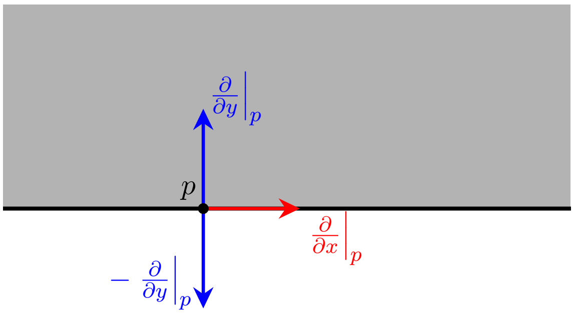
is C^∞. Thus, an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ induces an atlas $\left\{\left(U_\alpha \cap \partial M, \varphi_\alpha|_{U_\alpha \cap \partial M}\right)\right\}_{\alpha \in A}$ for ∂M, making ∂M into a C^∞ manifold of dimension $n - 1$ (without boundary).

§7.3 Tangent Vectors, Differential Forms, and Orientations

Let M be a manifold with boundary and $p \in \partial M$. Let us first understand what $C_p^\infty(M)$ is. Two C^∞ functions $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ defined on neighborhoods U and V of $p \in M$ and are said to be equivalent if f and g agree on some neighborhood W of p contained in $U \cap V$. It can easily be verified that the relation defined above is an equivalence relation. Under this equivalence relation, a germ of C^∞ functions at p is an equivalence class of such functions. With the usual pointwise addition, multiplication and scalar multiplication, the set $C_p^\infty(M)$ of germs of C^∞ functions at p is an $\mathbb{R}$-algebra.

The **tangent space** T_pM at p is then defined to be the vector space of all point derivations on the algebra $C_p^\infty(M)$ [Recall that a point derivation of $C_p^\infty(M)$ is a linear map $D_p : C_p^\infty(M) \to \mathbb{R}$ obeying Leibniz condition].

Let us now take the example where $p \in \partial\mathbb{H}^2$.



$\frac{\partial}{\partial x}\Big|_p$ and $\frac{\partial}{\partial y}\Big|_p$ are both point derivations on $C_p^\infty(\mathbb{H}^2)$. The tangent space $T_p\mathbb{H}^2$ is represented by a 2-dimensional vector space with origin at p and spanned by the tangent vectors $\frac{\partial}{\partial x}\Big|_p$ and $\frac{\partial}{\partial y}\Big|_p$. Since $T_p\mathbb{H}^2$ is a vector space and $\frac{\partial}{\partial y}\Big|_p \in T_p\mathbb{H}^2$, we have $-\frac{\partial}{\partial y}\Big|_p \in \mathbb{H}^2$.

The **cotangent space** T_p^*M to the point $p \in \partial M$ of the manifold M with boundary ∂M is defined to be the dual of the tangent space T_pM:

$$T_p^*M = \operatorname{Hom}(T_pM, \mathbb{R}). \tag{7.8}$$

By taking the disjoint union of the cotangent spaces T_p^*M for all points $p \in M$, i.e. over all interior and boundary points of M, one arrives at the cotangent bundle T^*M of the manifold with boundary. Define the vector bundle.

$$\Lambda^k(T^*M) = \bigsqcup_{P \in M} \Lambda^k\left(T_p^*M\right). \tag{7.9}$$

Then a **differential k-form** on M is a section of the vector bundle $\Lambda^k(T^*M)$. A differential k-form is C^∞ if it is C^∞ as a section of the vector bundle $\Lambda^k(T^*M)$. For example, $\mathrm{d}x \wedge \mathrm{d}y$ is a C^∞ 2-form on $\mathbb{H}^2$.

An **orientation** on an n-dimensional manifold M with boundary is again a C^∞ nowhere vanishing n-form on M. We've seen in Proposition 6.4 that the orientability of a manifold without boundary (or equivalently the existence of a C^∞ nowhere-vanishing top degree form by the definition of orientability of a manifold) is equivalent to the existence of an oriented atlas. The same goes for manifold with boundary.

In the proof of Proposition 6.4 for establishing the necessary and sufficient condition for the orientability of a manifold without boundary, it was necessary to replace the chart $(U, x^1, \dots, x^n)$ by $(U, -x^1, \dots, x^n)$. This would not be possible to carry out in the case $n = 1$ for manifold with boundary if we had not allowed the left-half line $\mathbb{L}^1$ as a local model in the definition of a chart on a 1-dimensional manifold with boundary. It would be better understood if we look at the following example.

Example 7.3. The closed interval $[0, 1]$ is a C^∞ manifold with boundary. It has an atlas with 2 charts (U_1, ϕ_1) and (U_2, ϕ_2), where $U_1 = [0, 1), \phi_1(x) = x$, and $U_2 = (0, 1]$, $\phi_2(x) = 1 - x$.

With $\mathrm{d}x$ as the orientation form, $[0,1]$ should be an oriented manifold with boundary. However, $\{(U_1, \phi_1), (U_2, \phi_2)\}$ is not an oriented atlas as the transition map

$$\left(\phi_2 \cdot \phi_1^{-1}\right)(x) = 1 - x$$

has negative Jacobian determinant on $\phi_1(0,1) = (0,1)$. If one flips the sign of ϕ_2, then $\{(U_1, \phi_1), (U_2, -\phi_2)\}$ becomes an oriented atlas as

$$\left(\phi_2 \cdot \phi_1^{-1}\right)(x) = x - 1$$

has positive Jacobian determinant on $\phi_1(0,1) = (0,1)$. It's important to note that $-\phi_2(x) = x - 1$ maps $U_2 = (0,1]$ to $(-1,0]$ being an open set of the left half-line $\mathbb{L}^1 = (-\infty, 0]$. If we had allowed only $\mathbb{H}^1$ as a local model for a 1-dimensional manifold with boundary, the closed interval $[0,1]$ wouldn't have an oriented atlas.

§7.4 Outward-Pointing Vector Fields

Definition 7.4. Let M be a manifold with boundary ∂M, and $p \in \partial M$. We say that a tangent vector $X_p \in T_pM$ is **inward-pointing** if $X_p \notin T_p(\partial M)$, and there are a positive real number ε and a curve $c : [0, \varepsilon) \to M$ such that $c(0) = p$, $c(0, \varepsilon) \in M^\circ$, and $c'(0) = X_p$. A vector $X_p \in T_pM$ is **outward-pointing** if $-X_p$ is inward-pointing.

For example, on the upper half-plane $\mathbb{H}^2$, the tangent vector $\left.\frac{\partial}{\partial y}\right|_p$ is inward-pointing, and $-\left.\frac{\partial}{\partial y}\right|_p$ is outward-pointing at $p \in \partial \mathbb{H}^2$.

A vector field along ∂M is a map that assigns to each point $p \in \partial M$, a tangent vector $X_p \in T_pM$ (as opposed to $T_p(\partial M)$). We say that a vector field X along the boundary ∂M is **outward-pointing** if for all $p \in \partial M$, $X_p \in T_pM$ is outward-pointing.

In a coordinate neighborhood $(U, x^1, \ldots, x^n)$ of p in M, a vector field X along ∂M can be written as a linear combination

$$X_q = \sum_i a^i(q) \left.\frac{\partial}{\partial x^i}\right|_q, \tag{7.10}$$

for $q \in \partial M$. The vector field X along ∂M is said to be smooth at $p \in M$ if there exists a coordinate neighborhood of p for which the functions a^i on ∂M are C^∞ at p; it is said to be smooth if it is smooth at every point p.

Lemma 7.4

Let M be a manifold with boundary and let $p \in \partial M$. Suppose $X_p \in T_pM$ is expressed as a linear combination of basis vectors on a chart $(U, x^1, \ldots, x^n)$ as follows:

$$X_p = \sum_i a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p.$$

Then X_p is outward pointing if and only if $a^n(p) < 0$.

Proof. ($\Rightarrow$) Suppose X_p is outward pointing, i.e. $Y_p = -X_p$ is inward pointing. Then $Y_p \notin T_p(\partial M)$ and there is a curve $c : [0, \varepsilon) \to M$ such that

$$c(0) = p,\ c'(0) = Y_p, \text{ and } c((0, \varepsilon)) \subseteq M^\circ.$$

Since $(U, \varphi) \equiv (U, x^1, \ldots, x^n)$ is a chart in the manifold with boundary M, U is diffeomorphic with an open subset of $\mathbb{H}^n$ via φ. Therefore, $x^n \geq 0$ on U. Since $p \in \partial M$, $x^n(p) = 0$. So if we take the curve $\varphi \circ c = (c^1, \ldots, c^n)$, where $c^i = x^i \circ c$, we have $c^n(0) = x^n(p) = 0$, and $c^n(t) \geq 0$ for $0 < t < \varepsilon$. Therefore,

$$\dot{c}^n(0) = \lim_{t \to 0^+} \frac{c^n(t) - c^n(0)}{t} = \lim_{t \to 0^+} \frac{c^n(t)}{t} \geq 0. \tag{7.11}$$

If $\dot{c}^n(0) = 0$, then $c^n(t) = 0$ for all $0 \le t \le \epsilon'$ for some $\epsilon' \le \varepsilon$. This would mean that

$$x^n(c(t)) = 0, \tag{7.12}$$

i.e. $c(t) \in \partial M$, which is not possible. Therefore, $\dot{c}^n(0) > 0$. Since $c'(0) = Y_p$, using *Proposition 6.3.1* of DG1,

$$Y_p = c'(0) = \sum_{i=1}^{n} \dot{c}^i(0) \left.\frac{\partial}{\partial x^i}\right|_p. \tag{7.13}$$

Since $Y_p = -X_p$, $-a^n(p) = \dot{c}^n(0) > 0$, i.e. $a^n(p) < 0$.

(⇐) Let $Y_p = -X_p$. By hypothesis, $-a^n(p) > 0$. We define $\alpha : \mathbb{R} \to \mathbb{R}^n$ as

$$\alpha(t) = \left(-a^1(p)\cdot t + p^1, -a^2(p)\cdot t + p^2, \ldots, -a^n(p)\cdot t + p^n\right), \tag{7.14}$$

where $\varphi(p) = (p^1, \ldots, p^n)$.

$p \in \partial M$, so $p^n = 0$. $\alpha(0) = \varphi(p) \in \varphi(U)$. Since $-a^n(p) > 0$, there exists $\varepsilon > 0$ such that $\alpha(t) \in \varphi(U) \subseteq \mathbb{H}^n$ for each $0 \le t < \varepsilon$. So we define the curve $c : [0, \varepsilon) \to U \subseteq M$ as

$$c(t) = \varphi^{-1}(\alpha(t)) = \varphi^{-1}\left(-a^1(p)\cdot t + p^1, -a^2(p)\cdot t + p^2, \ldots, -a^n(p)\cdot t\right). \tag{7.15}$$

Then clearly $c(0) = p$. For $0 < t < \varepsilon$, $-a^n(p)\cdot t > 0$, so $\alpha(t) \in (\mathbb{H}^n)^\circ$. As a result, $c(t) \in U^\circ \subseteq M^\circ$. Furthermore, $c'(0)$ is given by

$$\begin{aligned} c'(0) &= \sum_{i=1}^{n} \dot{c}^i(0) \left.\frac{\partial}{\partial x^i}\right|_p \\ &= \sum_{i=1}^{n} \frac{\mathrm{d}\left(-a^i(p)\cdot t + p^i\right)}{\mathrm{d}t}(0) \left.\frac{\partial}{\partial x^i}\right|_p \\ &= -\sum_i a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p = -X_p. \end{aligned} \tag{7.16}$$

Therefore, $-X_p$ is inward pointing, i.e. X_p is outward pointing. ■

Proposition 7.5

On a manifold M with boundary ∂M, there is a smooth outward-pointing vector field along ∂M.

Proof. Let $\{(U_\alpha, x_\alpha^1, x_\alpha^2, \ldots, x_\alpha^n)\}_{\alpha \in A}$ be an atlas for the manifold M. Let $\{\rho_\alpha\}_{\alpha \in A}$ be a partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha \in A}$. On each U_α, we take the vector field $-\frac{\partial}{\partial x_\alpha^n}$, and then we attach them using the partition of unity:

$$X = -\sum_\alpha \rho_\alpha \frac{\partial}{\partial x_\alpha^n}. \tag{7.17}$$

Now we show that X is an outward pointing vector field. First, we are going to show that X is smooth. Given any $q \in M$, there is a coordinate open set U that intersects only finitely many $\operatorname{supp}\rho_\alpha$'s due to the local finiteness of $\{\operatorname{supp}\rho_\alpha\}_\alpha$. Now, on the chart $(U, x^1, \ldots, x^n)$,

$$X = -\sum_\alpha \rho_\alpha \frac{\partial}{\partial x_\alpha^n} = -\sum_\alpha \rho_\alpha \sum_{i=1}^{n} \frac{\partial x^i}{\partial x_\alpha^n}\frac{\partial}{\partial x^i} = -\sum_{i=1}^{n}\sum_\alpha \rho_\alpha \frac{\partial x^i}{\partial x_\alpha^n}\frac{\partial}{\partial x^i}. \tag{7.18}$$

Here we swapped the order of summation, because they are finite sums. $\frac{\partial x^i}{\partial x_\alpha^n}$ is smooth, since the charts are C^∞-compatible. ρ_α are also smooth. Therefore, X is smooth on U. In particular, X is smooth at q. Since $q \in M$ was arbitrary, X is smooth on all of M.

Now we are going to show that X_p is outward-pointing for each $p \in \partial M$. There is a coordinate neighborhood V of p that intersects only finitely many $\operatorname{supp}\rho_\alpha$'s due to the local finiteness of $\{\operatorname{supp}\rho_\alpha\}_\alpha$. Suppose $(V, y^1, \dots, y^n)$ be a coordinate chart. Since $-\left.\frac{\partial}{\partial x_\alpha^n}\right|_p$ is an outward pointing vector in $T_p(V \cap U_\alpha)$, if we write

$$-\left.\frac{\partial}{\partial x_\alpha^n}\right|_p = \sum_{i=1}^n a_\alpha^i(p) \left.\frac{\partial}{\partial y^i}\right|_p, \tag{7.19}$$

we would have $a_\alpha^n(p) < 0$. Now, X_p is

$$X_p = -\sum_\alpha \rho_\alpha(p) \left.\frac{\partial}{\partial x_\alpha^n}\right|_p = \sum_\alpha \rho_\alpha(p) \sum_{i=1}^n a_\alpha^i(p) \left.\frac{\partial}{\partial y^i}\right|_p = \sum_{i=1}^n \sum_\alpha \rho_\alpha(p) a_\alpha^i(p) \left.\frac{\partial}{\partial y^i}\right|_p. \tag{7.20}$$

Here we swapped the order of summation, because they are finite sums. $\rho_\alpha(p) \geq 0$, and it is positive for at least one α since $\sum \rho_\alpha = 1$. Therefore, the coefficient of $\left.\frac{\partial}{\partial y^n}\right|_p$ in (7.20) is

$$\sum_\alpha \rho_\alpha(p) a_\alpha^n(p), \tag{7.21}$$

which is negative, since $a_\alpha^n(p) < 0$ for each α. Hence, X_p is outward pointing by Lemma 7.4. ■

§7.5 Interior Multiplication

If β is a k-covector on a vector space V, and $\mathbf{v} \in V$, for $k \geq 2$, the **interior multiplication** or **contraction** of β with $\mathbf{v}$ is the $(k-1)$-covector $\iota_\mathbf{v}\beta$ defined by

$$(\iota_\mathbf{v}\beta)(\mathbf{v}_2, \dots, \mathbf{v}_k) = \beta(\mathbf{v}, \mathbf{v}_2, \dots, \mathbf{v}_k), \tag{7.22}$$

with $\mathbf{v}_2, \dots, \mathbf{v}_k \in V$. When β is a 1-covector, then $\iota_\mathbf{v}\beta$ is supposed to be a constant real number (i.e. a 0-covector). This is defined by

$$\iota_\mathbf{v}\beta = \beta(\mathbf{v}) \in \mathbb{R}. \tag{7.23}$$

Finally, when β is a 0-covector on V (i.e. a constant real number), we define

$$\iota_\mathbf{v}\beta = 0. \tag{7.24}$$

Interior multiplication on a manifold is defined pointaise. If X is a smooth vector field on M and $\omega \in \Omega^k(M)$, then $\iota_X\omega \in \Omega^{k-1}(M)$, defined by

$$(\iota_X\omega)_p = \iota_{X_p}\omega_p. \tag{7.25}$$

The right side of (7.25) makes sense according to (7.22), (7.23), and (7.24). Indeed, for $(k-1)$ many tangent vectors $X_p^2, \dots, X_p^k$ with $k \geq 2$, (7.25) can be recast into the following using (7.22) as

$$(\iota_X\omega)_p\left(X_p^2, \dots, X_p^k\right) = (\iota_{X_p}\omega_p)\left(X_p^2, \dots, X_p^k\right) = \omega_p\left(X_p, X_p^2, \dots, X_p^k\right). \tag{7.26}$$

If $X, X^2, \dots, X^k$ are k-many smooth vector fields on M, then the RHS of (7.26) is $\left[\omega\left(X, X^2, \dots, X^k\right)\right](p)$ while the LHS of (7.26) is $\left[(\iota_X\omega)\left(X^2, \dots, X^k\right)\right](p)$. Therefore, for $k \geq 2$, one has

$$(\iota_X\omega)\left(X^2, \dots, X^k\right) = \omega\left(X, X^2, \dots, X^k\right), \tag{7.27}$$

for $(k-1)$ many C^∞ vector fields $X^2, \dots, X^k$ on M. Now, since ω is a smooth k-form, for any smooth vector fields $X, X^2, \dots, X^k$ on M, $\omega\left(X, X^2, \dots, X^k\right)$ is a smooth function on M. As a result, for any C^∞ vector fields $X^2, \dots, X^k$, $(\iota_X\omega)\left(X^2, \dots, X^k\right)$ is a smooth function on M. Therefore, $\iota_X\omega$ is a smooth $(k-1)$-form on M.

Proposition 7.6
For 1-covectors $\alpha^1, \ldots, \alpha^k$ on a vector space V and $\mathbf{v} \in V$,

$$\iota_{\mathbf{v}} \left(\alpha^1 \wedge \cdots \wedge \alpha^k\right) = \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(\mathbf{v})\, \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k,$$

where the caret $\widehat{}$ on the i-th covector α^i means that α^i is omitted from the wedge product.

Proof. Consider $k \geq 2$. For any $\mathbf{v}_2, \ldots, \mathbf{v}_k \in V$,

$$\begin{aligned}
\left[\iota_{\mathbf{v}} \left(\alpha^1 \wedge \cdots \wedge \alpha^k\right)\right] (\mathbf{v}_2, \ldots, \mathbf{v}_k) &= \left(\alpha^1 \wedge \cdots \wedge \alpha^k\right) (\mathbf{v}, \mathbf{v}_2, \ldots, \mathbf{v}_k) \\
&= \det \begin{bmatrix} \alpha^1(\mathbf{v}) & \alpha^1(\mathbf{v}_2) & \cdots & \alpha^1(\mathbf{v}_k) \\ \alpha^2(\mathbf{v}) & \alpha^2(\mathbf{v}_2) & \cdots & \alpha^2(\mathbf{v}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^k(\mathbf{v}) & \alpha^k(\mathbf{v}_2) & \cdots & \alpha^k(\mathbf{v}_k) \end{bmatrix} \\
&= \sum_{i=1}^{k} (-1)^{i+1} \alpha^i(\mathbf{v}) \det \left[\alpha^l(\mathbf{v}_j)\right]_{\substack{1 \leq l \leq k, l \neq i \\ 2 \leq j \leq k}} \\
&= \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(\mathbf{v}) \left(\alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k\right) (\mathbf{v}_2, \ldots, \mathbf{v}_k). \qquad (7.28)
\end{aligned}$$

Therefore,

$$\iota_{\mathbf{v}} \left(\alpha^1 \wedge \cdots \wedge \alpha^k\right) = \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(\mathbf{v})\, \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k. \qquad (7.29)$$

If $k = 1$,

$$\iota_{\mathbf{v}} \left(\alpha^1\right) = \alpha^1(\mathbf{v}) = (-1)^{1-1} \alpha^1(\mathbf{v}). \qquad (7.30)$$

So, the equality holds for $k = 1$ as well. ■

Lemma 7.7
The interior multiplication $\iota_X \omega$ of a smooth k-form ω on M with a smooth vector field X on M has the following properties:

(i) $\iota_{fX}\omega = f\, \iota_X \omega$;

(ii) $\iota_X (f\omega) = f\, \iota_X \omega$;

for $f \in C^\infty(M)$.

Proof. (i) Suppose $k \geq 2$. For any $p \in M$, and any $X_p^2, \ldots, X_p^k$,

$$\begin{aligned}
\left(\iota_{fX}\omega\right)_p \left(X_p^2, \ldots, X_p^k\right) &= \left(\iota_{f(p)X_p}\omega_p\right) \left(X_p^2, \ldots, X_p^k\right) \\
&= \omega_p \left(f(p) X_p, X_p^2, \ldots, X_p^k\right) \\
&= f(p) \omega_p \left(X_p, X_p^2, \ldots, X_p^k\right) \\
&= f(p) \left(\iota_X \omega\right)_p \left(X_p, X_p^2, \ldots, X_p^k\right). \qquad (7.31)
\end{aligned}$$

Since $p \in M$ is arbitrary, we have

$$\iota_{fX}\omega = f\, \iota_X \omega. \qquad (7.32)$$

Now consider the case $k = 1$.

$$\left(\iota_{fX}\omega\right)_p = \omega_p \left(f(p)\, X_p\right) = f(p) \left[\omega(X)\right]_p = \left(f\, \iota_X \omega\right)_p. \qquad (7.33)$$

Therefore, for $k = 1$ as well, since $p \in M$ is arbitrary,

$$\iota_{fX}\omega = f\,\iota_X\omega. \tag{7.34}$$

(ii) Again, let us first consider the case $k \geq 2$. For any $p \in M$, and any $X_p^2, \ldots, X_p^k$,

$$\begin{aligned}(\iota_X(f\omega))_p\left(X_p^2,\ldots,X_p^k\right) &= (\iota_{X_p}(f(p)\omega_p))\left(X_p^2,\ldots,X_p^k\right)\\ &= f(p)\omega_p\left(X_p, X_p^2,\ldots,X_p^k\right)\\ &= f(p)\ (\iota_X\omega)_p\left(X_p, X_p^2,\ldots,X_p^k\right).\end{aligned} \tag{7.35}$$

Since $p \in M$ is arbitrary, we have

$$\iota_X(f\omega) = f\,\iota_X\omega. \tag{7.36}$$

Now consider the case $k = 1$.

$$(\iota_X(f\omega))_p = \iota_{X_p}(f(p)\,\omega_p) = f(p)\,\omega_p(X_p) = f(p)\,[\omega(X)]_p = (f\,\iota_X\omega)_p\,. \tag{7.37}$$

Therefore, for $k = 1$ as well, since $p \in M$ is arbitrary,

$$\iota_X(f\omega) = f\,\iota_X\omega. \tag{7.38}$$

■

§7.6 Boundary Orientation

Now, we show that the boundary of an orientable manifold M with boundary is an orientable manifold without boundary.

Proposition 7.8

Let M be an orientable n-manifold with boundary ∂M. If ω is an orientation form on M and X is a smooth outward-pointing vector field along ∂M, then $\iota_X\omega$ is a smooth nowhere vanishing $(n-1)$ form on ∂M. Hence, ∂M is orientable.

Proof. Since ω is smooth on M, ω is also smooth on ∂M. By hypothesis, X is smooth on ∂M. Hence, the contraction $\iota_X\omega$ is also smooth on ∂M. We will now prove by contradiction that $\iota_X\omega$ is nowhere-vanishing on ∂M. Suppose. $\iota_X\omega$ vanishes at some $p \in \partial M$. It means that

$$(\iota_X\omega)_p\left(X_p^1,\ldots,X_p^{n-1}\right) = 0, \tag{7.39}$$

for any $X_p^1, \ldots, X_p^{n-1} \in T_p(\partial M)$. Let $Y_p^1, \ldots, Y_p^{n-1}$ be a basis for $T_p(\partial M)$. Since X is a smooth outward-pointing vector field along ∂M, $X_p \notin T_p(\partial M)$. Now,

$$\dim T_pM = \dim T_p(\partial M) + 1,$$

since ∂M is a manifold of dimension $n-1$ (without boundary). Since $X_p \notin T_p(\partial M)$ and $Y_p^1, \ldots, Y_p^{n-1}$ is a basis for $T_p(\partial M)$, one finds that $X_p, Y_p^1, \ldots, Y_p^{n-1}$ is a basis for T_pM. Hence,

$$\omega_p\left(X_p, Y_p^1,\ldots,Y_p^{n-1}\right) = (\iota_X\omega)_p\left(X_p^1,\ldots,X_p^{n-1}\right) = 0. \tag{7.40}$$

Now, by Lemma 6.1, since $X_p, Y_p' \cdots Y_p^{n-1}$ forms a basis for T_pM,

$$\omega_p\left(Z_p^1,\ldots,Z_p^n\right) = 0, \tag{7.41}$$

for all $Z_p^1, \ldots, Z_p^n \in T_pM$. In other words, $\omega_p \equiv 0$ on T_pM, a contradiction. ■

Example 7.4 (The boundary orientation on $\partial\mathbb{H}^n$)**.** An orientation form for the standard orientation on the upper half-space $\mathbb{H}^n$ is $\omega = \mathrm{d}x^1 \wedge \cdots \mathrm{d}x^n$. A smooth outward pointing vector field on $\partial\mathbb{H}^n$ is $-\frac{\partial}{\partial x^n}$. By Proposition 7.8, an orientation form on $\partial\mathbb{H}^n$ is given by the contraction

$$\iota_{-\frac{\partial}{\partial x^n}}(\omega) = \iota_{-\frac{\partial}{\partial x^n}}\left(\mathrm{d}x^1 \wedge \cdots \mathrm{d}x^n\right). \tag{7.42}$$

Using Proposition 7.6 and Lemma 7.7(i), we get

$$\begin{aligned}\iota_{-\frac{\partial}{\partial x^n}}(\omega) &= -\sum_{i=1}^{n}(-1)^{i-1}\left[\mathrm{d}x^i\left(\frac{\partial}{\partial x^n}\right)\right]\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \\ &= \sum_{i=1}^{n}(-1)^i\,\delta^i{}_n\,\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^{n-1} \wedge \mathrm{d}x^n \\ &= (-1)^n\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{n-1}.\end{aligned} \tag{7.43}$$

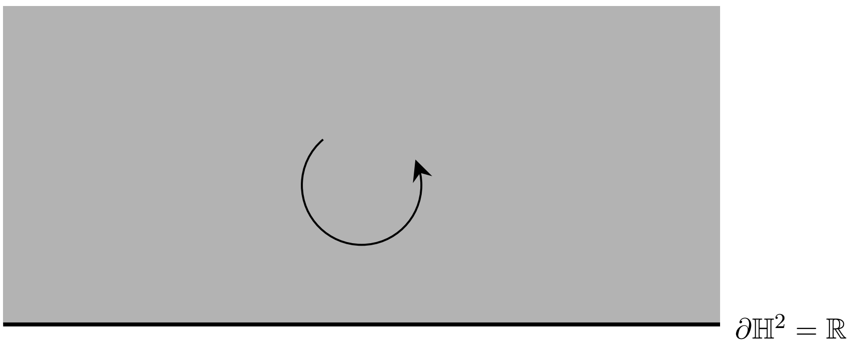
The $n = 1$ case needs a separate treatment. Recall that for a 1-covector β on a vector space V, and any vector $\mathbf{v} \in V$, the interior multiplication of β with $\mathbf{v}$ is defined as

$$\iota_{\mathbf{v}}\beta = \beta(\mathbf{v}) \in \mathbb{R}.$$

Now, for $\mathbb{H}^1 = [0, \infty)$, $\partial\mathbb{H}^1 = \{0\}$. Fix the orientation 1-form $\mathrm{d}x$ corresponding to the orientation directed from left to right. An outward pointing vector field on $\partial\mathbb{H}^1 = \{0\}$ is given by $-\frac{\partial}{\partial x}$. By Proposition 7.8, an orientation form on $\partial\mathbb{H}^1$ is given by the contraction

$$\iota_{-\frac{\partial}{\partial x}}(\mathrm{d}x) = -1. \tag{7.44}$$

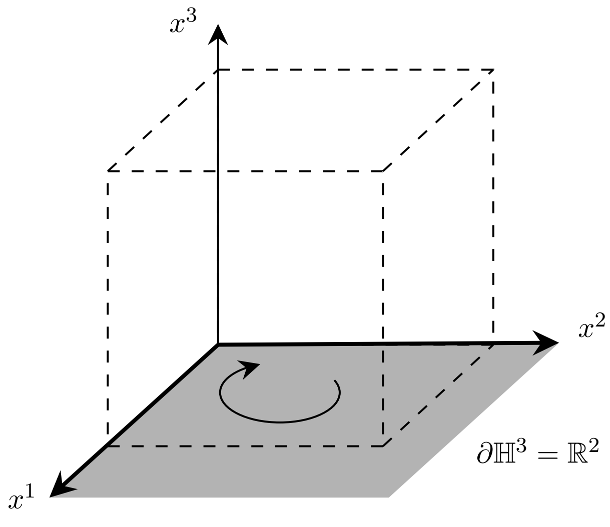
Let us now consider the case $n = 2$.



By (7.43), the boundary orientation on $\partial\mathbb{H}^2 = \mathbb{R}$ is given by the contraction

$$\iota_{-\frac{\partial}{\partial x^2}}\left(\mathrm{d}x^1 \wedge \mathrm{d}x^2\right) = \mathrm{d}x^1.. \tag{7.45}$$

Now consider $n = 3$.

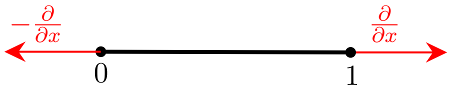


Here the manifold is $\mathbb{H}^3$ with boundary $\partial\mathbb{H}^3 = \mathbb{R}^2$, the x^1-x^2 plane (correspondint to $x^3 = 0$). According to (7.43), the boundary orientation on $\partial\mathbb{H}^3 = \mathbb{R}^2$ is given by the 2-form

$$(-1)^3 \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 = -\mathrm{d}x^1 \wedge \mathrm{d}x^2. \tag{7.46}$$

This yields the anti-clockwise orientation on the x^1-x^2 plane.

Example 7.5. Consider the closed interval $[0,1]$ in $\mathbb{R}$. One has $\partial\,[0,1] = \{0,1\}$. Also consider the orientation 1-form $\mathrm{d}x$ on $[0,1]$ corresponding to the standard orientation directed from left to right.



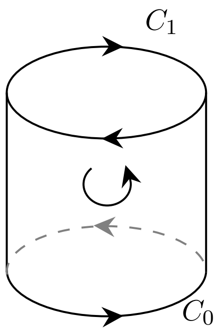
At the right boundary point 1, an outward pointing vector field reads $\frac{\partial}{\partial x}$. Hence, by Proposition 7.8, the boundary orientation at 1 is given by

$$\iota_{\frac{\partial}{\partial x}}(\mathrm{d}x) = 1. \tag{7.47}$$

On the other hand, at the left boundary point 0, an outward pointing vector field reads $-\frac{\partial}{\partial x}$. Hence, by Proposition 7.8, the boundary orientation at 0 is given by the contraction

$$\iota_{-\frac{\partial}{\partial x}}(\mathrm{d}x) = -1. \tag{7.48}$$

Example 7.6. Let M be the cylinder $S^1 \times [0,1]$ with the counterclockise orientation when viewed from the exterior. Let us determine the boundary orientation on $C_0 = S^1 \times \{0\}$ and $C_1 = S^1 \times \{1\}$.



The counterclock arise orientation on M is given by the orientation form $\omega = \mathrm{d}\theta \wedge \mathrm{d}t$. An outward-pointing vector field on C_0 is given by $-\frac{\partial}{\partial t}$, so that the relevant contraction of ω with $-\frac{\partial}{\partial t}$ reads

$$\iota_{-\frac{\partial}{\partial t}}(\mathrm{d}\theta \wedge \mathrm{d}t) = -\frac{\partial}{\partial t}(\mathrm{d}t)\,(-1)^{2-1}\,\mathrm{d}\theta = \mathrm{d}\theta. \tag{7.49}$$

Hence, the boundary orientation on C_0 is given by the 1-form $\mathrm{d}\theta$.

Now, to determine the boundary orientation on $C_1 = S^1 \times \{1\}$, let us compute the contraction of ω on with an outward-pointing vector field $\frac{\partial}{\partial t}$ on C_1 :

$$\iota_{\frac{\partial}{\partial t}}(\mathrm{d}\theta \wedge \mathrm{d}t) = \frac{\partial}{\partial t}(\mathrm{d}t)\,(-1)^{2-1}\,\mathrm{d}\theta = -\mathrm{d}\theta. \tag{7.50}$$

So the boundary orientation on C_1 is given by the 1-form $-\mathrm{d}\theta$. Therefore, on C_0, the orientation is given by the counterclockwise orientation and on C_1, the orientation is given by the clockwise orientation viewed from the top.

Remark 7.3. Recall the nowhere-vanishing 1 form on S^1 from Example 5.2:

$$\omega = \begin{cases} \frac{\mathrm{d}y}{x} & \text{on } U_x = \{(x,y) \in S^1 \mid x \neq 0\}, \\ -\frac{\mathrm{d}x}{y} & \text{on } U_y = \{(x,y) \in S^1 \mid y \neq 0\}. \end{cases} \tag{7.51}$$

If we go back to polar coordinates, i.e. $x = \cos\theta$ and $y = \sin\theta$, then

$$\frac{\mathrm{d}y}{x} = \frac{\mathrm{d}(\sin\theta)}{\cos\theta} = \frac{\cos\theta\,\mathrm{d}\theta}{\cos\theta} = \mathrm{d}\theta, \tag{7.52}$$

$$-\frac{\mathrm{d}x}{y} = -\frac{\mathrm{d}(\cos\theta)}{\sin\theta} = -\frac{-\sin\theta\,\mathrm{d}\theta}{\sin\theta} = \mathrm{d}\theta. \tag{7.53}$$

Therefore, this nowhere vanishing 1-form ω is, in fact, $\mathrm{d}\theta$.

8 Integration on Manifolds

§8.1 Riemann Integral Review

Let us first recall the subject of Riemann integration over a closed rectangle in Euclidean space $\mathbb{R}^n$. A closed rectangle in $\mathbb{R}^n$ is a cartesian product

$$R = \left[a^1, b^1\right] \times \ldots \times [a^n, b^n]$$

of closed intervals in $\mathbb{R}$ with $a^i, b^i \in \mathbb{R}$. Let f be a bounded function $f : R \to \mathbb{R}$ defined on a closed rectangle R. The volume vol(R) of the closed rectangle is defined to be

$$\text{vol}\,(R) = \prod_{i=1}^{n} (b_i - a_i)\,. \tag{8.1}$$

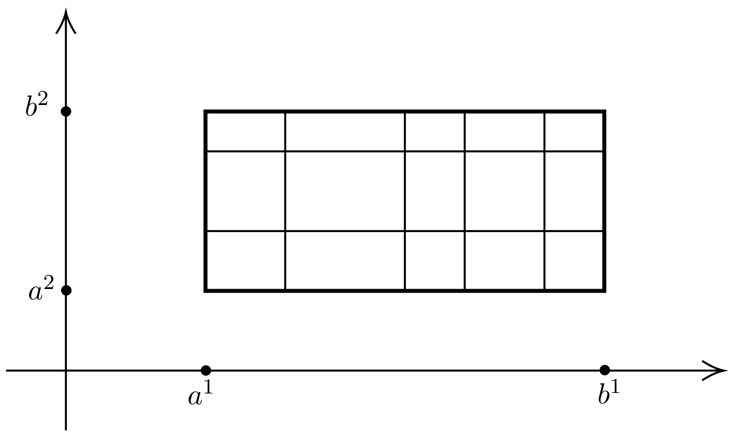
A partition of the closed interval $[a, b]$ is a set of real numbers $\{p_0, \ldots, p_n\}$ arranged in ascending order, i.e.

$$a = p_0 < p_1 < \ldots < p_n = b \tag{8.2}$$

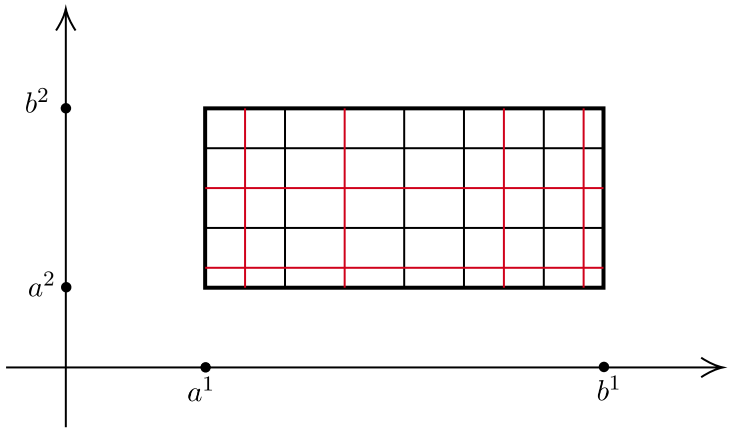
A partition of the rectangle R is a collection $P = \{P_1, \ldots, P_n\}$ such that each P_i is a partition of $\left[a^i, b^i\right]$. In other words. each P_i is an increasing sequence of real numbers

$$a^i_i = p^i_0 < p^i_1 < \ldots\ldots < p^i_{k^i} = b^i \tag{8.3}$$

for $i = 1, 2, \ldots\ldots n$ This way, the partition P of the closed rectangle R divides it into closed subrectangles, which we denote by R_j. On possible partition of a closed rectangle $[a^1, b^1] \times [a^2, b^2]$ is pictured below in $\mathbb{R}^2$:



A partition $P' = \{P'_1, \ldots, P'_n\}$ of the same rectangle R is called a **refinement** of the partition $P = \{P_1, \ldots, P_n\}$ of R if $P_i \subset P'_i$ for each $i = 1, 2, \ldots, n$. For example, the following partition of $[a^1, b^1] \times [a^2, b^2]$ is a refinement of the partition shown above. (The original partition P is drawn in black, while the new lines arising in the refined partition are drawn in red.)



We immediately see that each subrectangle R_j of P is subdivided into subrectangles R'_{jk} of the refinement P'. It's now time to define the lower and upper sum of the bounded function $f : R \to \mathbb{R}$ wirh respect to the partition P:

$$L(f, P) := \sum_{R_j} \left(\inf_{R_j} f\right) \mathrm{vol}(R_j), \tag{8.4}$$

$$U(f, P) := \sum_{R_j} \left(\sup_{R_j} f\right) \mathrm{vol}(R_j). \tag{8.5}$$

It's clear that for any partition P,

$$L(f, P) \leq U(f, P). \tag{8.6}$$

Now, suppose P' is a refinement of the partition P. Then each subrectangle R_j of P is subdivided into subrectangles R'_{jk} of the refinement P'. Furthermore, $\mathrm{vol}(R_j) = \sum_k \mathrm{vol}\left(R'_{jk}\right)$. Now, since $R'_{jk} \subset R_j$, one has

$$\inf_{R_j} f \leq \inf_{R'_{jk}} f \text{ and } \sup_{R_j} f \geq \sup_{R'_{jk}} f. \tag{8.7}$$

Now, from (8.4),

$$\begin{aligned} L(f, P) &= \sum_{R_j} \left(\inf_{R_j} f\right) \mathrm{vol}(R_j) \\ &\leq \sum_{R_j} \left(\inf_{R'_{jk}} f\right) \sum_k \mathrm{vol}\left(R'_{jk}\right) \\ &\leq \sum_{R'_{jk}} \left(\inf_{R'_{jk}} f\right) \mathrm{vol}\left(R'_{jk}\right) = L(f, P'). \end{aligned}$$

In other words,

$$L(f, P) \leq L(f, P'). \tag{8.8}$$

Similarly,

$$U(f, P) \geq U(f, P'). \tag{8.9}$$

Any two partitions P and P' of the rectangle R have a common refinement $Q = \{Q_1, \ldots, Q_n\}$ with $Q_i = P_i \cup P'_i$. Then by (8.8) and (8.9),

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P').$$

Therefore, for any two partitions P and P',

$$L(f, P) \leq U(f, P'). \tag{8.10}$$

From (8.10), one sees that $U(f,P')$ is an upper bound of $L(f,P)$ for any partition P of the rectangle R. As a result,

$$\sup_P L(f,P) \le U(f,P') . \tag{8.11}$$

Again, from (8.11), one sees that $\sup_P L(f,P)$ is a lower bound of $U(f,P')$ for any partition P' of the rectangle R. Hence,

$$\sup_P L(f,P) \le \inf_{P'} U(f,P') . \tag{8.12}$$

The supremum of the lower-sum $L(f,P)$ as P varies over all paritions of the rectangle is called the **lower integral**, and is denoted by

$$\underline{\int}_R f := \sup_P L(f,P) \tag{8.13}$$

On the contrary, the infimum of the upper-sum $U(f,P)$ as P varies over all paritions of the rectangle is called the **upper integral**, and is denoted by

$$\overline{\int}_R f := \inf_P U(f,P) . \tag{8.14}$$

Using these notations (8.12) reads

$$\underline{\int}_R f \le \overline{\int}_R f. \tag{8.15}$$

Definition 8.1 (Riemann integrability). Let R be a closed rectangle in $\mathbb{R}^n$. A bounded function $f : R \to \mathbb{R}$ is said to be **Riemann integrable** if $\underline{\int}_R f = \overline{\int}_R f$. In this case, the **Riemann integral** of f is this common value, usually denoted by

$$\int_R f(\mathbf{x})\, \mathrm{d}x^1 \mathrm{d}x^2 \cdots \mathrm{d}x^n,$$

where $x^1, \ldots, x^n$ are the standard coordinates on $\mathbb{R}^n$.

Definition 8.2. If $f : A \subset \mathbb{R}^n \to R$, then the **extension of f by zero** is the function $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

If f is a bounded function on a bounded set A, then one encloses A in a closed rectangle $R \subset \mathbb{R}^n$. The the Riemann integral of $f : A \to \mathbb{R}$ over A is defined to be

$$\int_A f(\mathbf{x})\, \mathrm{d}x^1 \mathrm{d}x^2 \cdots \mathrm{d}x^n := \int_R \overline{f}(\mathbf{x})\, \mathrm{d}x^1 \mathrm{d}x^2 \cdots \mathrm{d}x^n, \tag{8.16}$$

provided the RHS exists, i.e. the extension $\overline{f}$ of f by zero is Riemann integrable over the closed rectangle R enclosing A. The **volume** $\operatorname{vol} A$ of a subset $A \subset \mathbb{R}^n$ is defined to be the integral $\int_A 1\, \mathrm{d}x^1 \mathrm{d}x^2 \cdots \mathrm{d}x^n$ if the integral exists.

Integrability conditions

Definition 8.3. A set $A \subset \mathbb{R}^n$ is said to have **measure zero** if for every $\varepsilon > 0$, there is a countable collection of closed rectangles $\{R_i\}_{i=1}^{\infty}$ such that $A \subset \bigcup_{i=1}^{\infty} R_i$ and

$$\sum_{i=1}^{\infty} \operatorname{vol}(R_i) < \varepsilon.$$

Theorem 8.1 (Lebesgue's theorem)

A bounded function $f : A \to \mathbb{R}$ on a bounded subset $A \subset \mathbb{R}^n$ is Riemann integrable if and only if the set $\operatorname{Disc}\left(\overline{f}\right)$ of discontinuities of the extended function $\overline{f}$ has measure zero.

Proposition 8.2

If a continuous function $f : U \to \mathbb{R}$ defined on an open subset U of $\mathbb{R}^n$ has compact support, then f is Riemann integrable on U.

Proof. $f : U \to \mathbb{R}$ is continuous. On $U \setminus \operatorname{supp} f$, f is zero. Since f is continuous on $\operatorname{supp} f \subset U$, and $\operatorname{supp} f$ is compact (by hypothesis), f is bounded on $\operatorname{supp} f$. Also, f is zero on $U \setminus \operatorname{supp} f$. Hence, $f : U \to \mathbb{R}$ is a bounded continuous function on the open subset $U \subset \mathbb{R}^n$. We claim that the extension $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ is continuous.

By the definition of extension of a function by zero, $\overline{f}$ agrees witt f on U, and hence $\overline{f}$ is continuous on U. It remains to show that f is continuous on the complement $\mathbb{R}^n \setminus U$ of U. Since $\operatorname{supp} f \subset U$, if $p \notin U$, then $P \notin \operatorname{supp} f$.

$\operatorname{supp} f$ being a compact subset of $\mathbb{R}^n$ is closed and bounded. Hence. $\mathbb{R}^n \setminus \operatorname{supp} f$ is open and $p \in \mathbb{R}^n \setminus \operatorname{supp} f$. Therefore, there exist an open ball B such that

$$p \in B \subset \mathbb{R}^n \setminus \operatorname{supp} f,$$

i.e. an open ball containing p and disjoint from $\operatorname{supp} f$. On this open ball B containing p, $f \equiv 0$. Therefore, f is

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases} \tag{8.17}$$

On $B \setminus U$, $\overline{f}$ is clearly 0 . Since $B \cap U \cap \operatorname{supp} f = \varnothing$, and on $B \cap U$, f and $\overline{f}$ agree with other, one must have, $\overline{f} = 0$ on $B \cap U$. Hence, on B, $\overline{f} \equiv 0$. This implies that $\overline{f}$ is continuous at $p \in U$. We, therefore, have, $\overline{f}$ to be continuous on the whole of $\mathbb{R}^n$.

Note that $\overline{f}$, defined by (8.17) is also the zero extension of $f|_{\operatorname{supp} f} : \operatorname{supp} f \to \mathbb{R}$ with $\operatorname{supp} f$ being a bounded subset of $\mathbb{R}^n$. Now, we are all good to apply Lebesgue's theorem by which $f|_{\operatorname{supp} f} : \operatorname{supp} f \to \mathbb{R}$ is Riemann integravle integrable. Since f is zero on $U \setminus \operatorname{supp} f$, $f : U \to \mathbb{R}$ is also Riemann integrable. ■

Definition 8.4 (Domain of integration). A subset $A \subset \mathbb{R}^n$ is called a **domain of integration** if it is bounded and its topological boundary $\operatorname{bd} A$ is a set of measure zero.

Familiar plane figures, such as triangles, rectangles, disks are all domains of integration in $\mathbb{R}^2$.

Proposition 8.3

Every bounded continuous function f defined on a domain of integration A in $\mathbb{R}^n$ is Riemann integrable over A.

Proof. Let $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ be the extension of f by zero, ie.,

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \notin U. \end{cases} \tag{8.18}$$

Since f is continuous on A by hypothesis, $\overline{A}$ is necessarily continuous in the open set $\text{int}(A)$. One also observes that if p is an exterior point of A, i.e., if $p \in \mathbb{R}^n \setminus \overline{A}$, being an open set, there is an open set $U \subset \mathbb{R}^n$ such that

$$p \in U \subseteq \mathbb{R}^n \setminus \overline{A}.$$

Since $U \cap A = \varnothing$, $\overline{f} \equiv 0$ on U. Hence, $\overline{f}$ is continuous at p. One, thus, verifies that $\overline{f}$ is continuous at all interior and exterior points of A. Therefore, the set $\text{Disc}\left(\overline{f}\right)$ of discontinuities of $\overline{f}$ is a subset of $\text{bd}\,(A)$, a set of measure zero. By Lebergues' theorem, f is Riemann integrable on A. ■

§8.2 Integral of an n-form on $\mathbb{R}^n$

Definition 8.5. Let $\omega = f(\mathbf{x})\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ be a C^∞ n-form on an open subset $U \subset \mathbb{R}^n$, with standard coordinates $x^1, \ldots, x^n$. Its integral over a subset $A \subset U$ is defined to be the Riemann integral of $f(\mathbf{x})$

$$\int_A \omega = \int_A f(\mathbf{x})\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n := \int_A f(\mathbf{x})\, \mathrm{d}x^1 \cdots \mathrm{d}x^n, \tag{8.19}$$

if the Riemann integral exists.

If f is a bounded continuous function on a domain of integration A in $\mathbb{R}^n$, then the integral $\int_A f\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ exists by Proposition 8.3.

Let us now see how the integral of an n-form $\omega = f\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ on an open subset $U \subseteq \mathbb{R}^n$ transform under change of variables. A change of variables on $U \subseteq \mathbb{R}^n$ is given by a diffeomorphism

$$T : V \subseteq \mathbb{R}^n \to U \subseteq \mathbb{R}^n.$$

Let $x^1, \ldots, x^n$ be the standard coordinates on U and $y^1, \ldots, y^n$ the standard coordinates on V. One, therefore, has $\left\{ \frac{\partial}{\partial x^1}\Big|_{T(p)}, \ldots, \frac{\partial}{\partial x^n}\Big|_{T(p)} \right\}$ to be a basis of $T_{T(p)}\mathbb{R}^n$ while $\left\{ \frac{\partial}{\partial y^1}\Big|_p, \ldots, \frac{\partial}{\partial y^n}\Big|_p \right\}$ is a basis of $T_p\mathbb{R}^n$. for $p \in V$. Now, the differential $DT(p) : T_p\mathbb{R}^n \to T_{T(p)}\mathbb{R}^n$ at $p \in V$ is represented by the following $n \times n$ matrix:

$$DT(p) = \begin{bmatrix} \frac{\partial T^1}{\partial y^1}(p) & \frac{\partial T^1}{\partial y^2}(p) & \cdots & \frac{\partial T^1}{\partial y^n}(p) \\ \frac{\partial T^2}{\partial y^1}(p) & \frac{\partial T^2}{\partial y^2}(p) & \cdots & \frac{\partial T^2}{\partial y^n}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial T^n}{\partial y^1}(p) & \frac{\partial T^n}{\partial y^2}(p) & \cdots & \frac{\partial T^n}{\partial y^n}(p) \end{bmatrix}$$

The deterimant of the matrix DT is precisely the Jacobian determinant denoted by $\det(J(T))$, i.e. $\det(J(T)) = \det(DT)$, that arises in the change of variable formula for integration in multivariable calculus:

$$\int_U f\, \mathrm{d}x^1 \cdots \mathrm{d}x^n = \int_V (f \circ T)\, |\det(DT)|\, \mathrm{d}y^1 \cdots \mathrm{d}y^n, \tag{8.20}$$

with $f : U \subset \mathbb{R}^n \to \mathbb{R}$ being a bounded continuous function and $f \circ T : V \subset \mathbb{R}^n \to \mathbb{R}$. Here, we assume that U and V are both connected. By Lemma 5.5, one has

$$\mathrm{d}T^1 \wedge \cdots \mathrm{d}T^n = \det\left[\frac{\partial T^i}{\partial y^j}\right] \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n, \tag{8.21}$$

where $T^i = x^i \circ T = T^*\left(x^i\right)$ is the i-th component of T. Hence, for $T : V \subset \mathbb{R}^n \to U \subset \mathbb{R}^n$ and ω

being an n-form on U,

$$\begin{aligned}\int_V T^*\omega &= \int_V T^*\left(f\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\right)\\ &= \int_V (T^*f)\,T^*\mathrm{d}x'\wedge\cdots\wedge T^*\mathrm{d}x^n\\ &= \int_V (f\circ T)\,\mathrm{d}\left(T^*x^1\right)\wedge\cdots\wedge\mathrm{d}\left(T^*x^n\right)\\ &= \int_V (f\circ T)\,\mathrm{d}T^1\wedge\cdots\wedge\mathrm{d}T^n\\ &= \int_V (f_0T)\det(J(T))\,\mathrm{d}y^1\wedge\cdots\wedge\mathrm{d}y^n. \end{aligned}\tag{8.22}$$

Using (8.20),

$$\int_U\omega = \int_U f\,\mathrm{d}x^1\cdots\mathrm{d}x^n = \int_V (f\circ T)\left|\det J(T)\right|\,\mathrm{d}y^1\cdots\mathrm{d}y^n.\tag{8.23}$$

Since (8.22) and (8.23) differ by the sign of $\det(J(T))$, one has

$$\int_V T^*\omega = \pm\int_U\omega,\tag{8.24}$$

depending on whether the Jacobian determinant $\det(J(T))$ is positive or negative. By Proposition 6.3, a diffeomorphism $T: V\subseteq\mathbb{R}^n\to U\in\mathbb{R}^n$ is orientation-preserving if and only if its Jacobian determinant

$$\det J(T) = \det\left[\frac{\partial T^i}{\partial y^j}\right]$$

is everywhere positive on V. Equation (8.24) tells us that the integral of differential form ω is not necessarily invariant under an arbitrary diffeomorphism $T: V\to U$. The integral of a differential form ω is only invariant ($\int_V T^*\omega = \int_U\omega$) if and only if the diffeomorphism $T: V\to U$ is orientation preserving.

§8.3 Integral of a differential form over a manifold

Our approach to integration over a general manifold has the following distinguishing features:

(a) The manifold must be oriented.

(b) On a manifold of dimension n, one can only integrate n-forms, not functions (which are 0-forms).

(c) The n-forms must have compact support.

Let M be an oriented manifold of dimension n, with an oriented atlas $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in A}$ giving the orientation of M. If $\omega\in\Omega^k(M)$, then

$$\operatorname{supp}\omega = \overline{\{p\in M\mid\omega_p\neq 0\}} = \mathrm{cl}_M\left(\{p\in M\mid\omega_p\neq 0\}\right).\tag{8.25}$$

Lemma 8.4

If (U,φ) is a chart in a manifold M (of dimension n) and ω is an n-form on U,

$$\operatorname{supp}\left[\left(\varphi^{-1}\right)^*\omega\right] = \varphi\left(\operatorname{supp}\omega\right).$$

Proof. Here $\varphi : U \to \varphi(U)$ is a diffeomorphism. In particular, it is a homeomorphism. Therefore. $\varphi\left(\overline{A}\right) = \overline{\varphi(A)}$. Now,

$$\begin{aligned}
\operatorname{supp}\left[\left(\varphi^{-1}\right)^{*}\omega\right] &= \overline{\left\{q = \varphi(p) \in \varphi(U) \mid \left[\left(\varphi^{-1}\right)^{*}\omega\right]_q \neq 0\right\}} \\
&= \overline{\varphi\left(\left\{p \in U \mid \left[\left(\varphi^{-1}\right)^{*}\omega\right]_{\varphi(p)} \neq 0\right\}\right)} \\
&= \varphi\left(\overline{\left\{p \in U \mid \left[\left(\varphi^{-1}\right)^{*}\omega\right]_{\varphi(p)} \neq 0\right\}}\right).
\end{aligned} \tag{8.26}$$

Now, $\left[\left(\varphi^{-1}\right)^{*}\omega\right]_{\varphi(p)} \in \Lambda^k\left(T^*_{\varphi(p)}\varphi(U)\right)$. It is 0 if and only if it yields 0 when applied to any basis vectors. Therefore,

$$\begin{aligned}
\left[\left(\varphi^{-1}\right)^{*}\omega\right]_{\varphi(p)} = 0 &\iff \left[\left(\varphi^{-1}\right)^{*}\omega\right]_{\varphi(p)}\left(\left.\frac{\partial}{\partial r^{i_1}}\right|_{\varphi(p)}, \dots, \left.\frac{\partial}{\partial r^{i_k}}\right|_{\varphi(p)}\right) = 0 \text{ for all } I = (i_1, \dots, i_k) \\
&\iff \omega_{\varphi^{-1}(\varphi(p))}\left(\left(\varphi^{-1}\right)_{*,\varphi(p)}\left.\frac{\partial}{\partial r^{i_1}}\right|_{\varphi(p)}, \dots, \left(\varphi^{-1}\right)_{*,\varphi(p)}\left.\frac{\partial}{\partial r^{i_k}}\right|_{\varphi(p)}\right) = 0 \\
&\qquad\qquad \text{for all } I = (i_1, \dots, i_k) \\
&\iff \omega_p\left(\left.\frac{\partial}{\partial x^{i_1}}\right|_p, \dots, \left.\frac{\partial}{\partial x^{i_k}}\right|_p\right) = 0 \text{ for all } I = (i_1, \dots, i_k) \\
&\iff \omega_p = 0.
\end{aligned} \tag{8.27}$$

Now using (8.26), we get

$$\begin{aligned}
\operatorname{supp}\left[\left(\varphi^{-1}\right)^{*}\omega\right] &= \varphi\left(\overline{\left\{p \in U \mid \left[\left(\varphi^{-1}\right)^{*}\omega\right]_{\varphi(p)} \neq 0\right\}}\right) \\
&= \varphi\left(\overline{\{p \in U \mid \omega_p \neq 0\}}\right) \\
&= \varphi(\operatorname{supp}\omega).
\end{aligned} \tag{8.28}$$

■

Lemma 8.5

If $\omega, \tau \in \Omega^*(M)$, then

(a) $\operatorname{supp}(\omega + \tau) \subseteq \operatorname{supp}\omega \cup \operatorname{supp}\tau$.

(b) $\operatorname{supp}(\omega \wedge \tau) \subseteq \operatorname{supp}\omega \cap \operatorname{supp}\tau$.

Proof. (a) If $(\omega + \tau)_p \neq 0$, then $\omega_p \neq 0$ or $\tau_p \neq 0$. Therefore,

$$\left\{p \in M \mid (\omega + \tau)_p \neq 0\right\} \subseteq \{p \in M \mid \omega_p \neq 0\} \cup \{p \in M \mid \tau_p \neq 0\}. \tag{8.29}$$

Taking closure on both sides, and using the fact that $\overline{A \cup B} = \overline{A} \cup \overline{B}$, we get

$$\overline{\left\{p \in M \mid (\omega + \tau)_p \neq 0\right\}} \subseteq \overline{\{p \in M \mid \omega_p \neq 0\}} \cup \overline{\{p \in M \mid \tau_p \neq 0\}}. \tag{8.30}$$

In other words,

$$\operatorname{supp}(\omega + \tau) \subseteq \operatorname{supp}\omega \cup \operatorname{supp}\tau. \tag{8.31}$$

(b) If $(\omega \wedge \tau)_p \neq 0$, then $\omega_p \neq 0$ and $\tau_p \neq 0$. Therefore,

$$\left\{p \in M \mid (\omega \wedge \tau)_p \neq 0\right\} \subseteq \{p \in M \mid \omega_p \neq 0\} \cap \{p \in M \mid \tau_p \neq 0\}. \tag{8.32}$$

Taking closure on both sides, and using the fact that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$, we get

$$\overline{\left\{p \in M \mid (\omega \wedge \tau)_p \neq 0\right\}} \subseteq \overline{\{p \in M \mid \omega_p \neq 0\}} \cap \overline{\{p \in M \mid \tau_p \neq 0\}}. \tag{8.33}$$

In other words,

$$\operatorname{supp}(\omega \wedge \tau) \subseteq \operatorname{supp} \omega \cap \operatorname{supp} \tau. \tag{8.34}$$

■

Let $\Omega_c^k(M)$ denote the vector space of C^∞ k-forms with compact support on M. Suppose (U, φ) is a chart in the atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$.

If $\omega \in \Omega_c^n(U)$ is an n-form with compact support on U, since φ being a diffeomorphism is continuous, $\varphi(\operatorname{supp} \omega)$ is compact in $\varphi(U)$. Then by Lemma 8.4, supp $\left[(\varphi^{-1})^* \omega\right]$ is compact in $\varphi(U) \subseteq \mathbb{R}^n$. We define the integral of ω on U by

$$\int_U \omega = \int_{\varphi(u) \subset \mathbb{R}^n} \left(\varphi^{-1}\right)^* \omega. \tag{8.35}$$

If (U, ψ) is another chart in the oriented atlas with the same open set U, then $\varphi \circ \psi^{-1} : \psi(U) \to \varphi(U)$ is an orientation preserving diffeomorphism (i.e. with positive Jacobian determinant), so it preserves integral of n-form on open subset of $\mathbb{R}^n$. Therefore,

$$\begin{aligned}
\int_{\varphi(U)} \left(\varphi^{-1}\right)^* \omega &= \int_{\psi(U)} \left(\varphi \circ \psi^{-1}\right)^* \left[\left(\varphi^{-1}\right)^* \omega\right] \\
&= \int_{\psi(U)} \left[\left(\psi^{-1}\right)^* \circ \varphi^* \circ \left(\varphi^{-1}\right)^*\right] \omega \\
&= \int_{\psi(U)} \left[\left(\psi^{-1}\right)^* \circ \left(\varphi^{-1} \circ \varphi\right)^*\right] \omega \\
&= \int_{\psi(U)} \left(\psi^{-1}\right)^* \omega,
\end{aligned} \tag{8.36}$$

proving the chart independence of the definition (8.35). By the linearity of integral on $\mathbb{R}^n$ and linearity of pullback, if $\omega, \tau \in \Omega_c^n(U)$, then

$$\int_U (\omega + \tau) = \int_U \omega + \int_U \tau. \tag{8.37}$$

Now, let $\omega \in \Omega_c^n(M)$. Choose a partition of unity $\{\rho_\alpha\}_\alpha$ subordinate to the open cover $\{U_\alpha\}_\alpha$. From the definition of partition of unity $\{\operatorname{supp} \rho_\alpha\}_\alpha$ is locally finite. Let $p \in \operatorname{supp} \omega$. There is a neighborhood W_p of p in M that intersects only finitely many of the sets $\operatorname{supp} \rho_\alpha$ (from the local finiteness of the set $\{\operatorname{supp} \rho_\alpha\}_\alpha$). The collection $\{W_p \mid p \in \operatorname{supp} \omega\}$ obviously covers $\operatorname{supp} \omega$. Since $\operatorname{supp} \omega$ is compact, there is a finite subcover of $\{W_p \mid p \in \operatorname{supp} \omega\}$ of $\operatorname{supp} \omega$. Let us denote this subcover by $\{W_{p_1}, \ldots, W_{p_m}\}$. In other words,

$$\operatorname{supp} \omega \subseteq \bigcup_{i=1}^m W_{p_i} \tag{8.38}$$

Since each W_{p_i} intersects finitely many $\operatorname{supp} \rho_\alpha$ in $\{\operatorname{supp} \rho_\alpha\}_\alpha$, $\operatorname{supp} \omega$ must intersect only finitely many $\operatorname{supp} \rho_\alpha$. By Lemma 8.5(b),

$$\operatorname{supp}(\rho_\alpha \omega) \subseteq \operatorname{supp} \rho_\alpha \cap \operatorname{supp} \omega. \tag{8.39}$$

Thus for all but finitely many α, $\operatorname{supp}(\rho_\alpha \omega)$ is empty, i.e., $\rho_\alpha \omega \equiv 0$. Therefore, $\sum_\alpha \rho_\alpha \omega$ is a **finite** sum, and

$$\sum_\alpha \rho_\alpha \omega = \omega, \tag{8.40}$$

since $\sum_\alpha \rho_\alpha = 1$. By (8.39), $\operatorname{supp}(\rho_\alpha \omega) \subseteq \operatorname{supp} \omega$, i.e. $\operatorname{supp}(\rho_\alpha \omega)$ is a closed subset of a compact set $\operatorname{supp} \omega$. Hence, $\operatorname{supp}(\rho_\alpha \omega)$ is compact. As a result, $\rho_\alpha \omega$ is an n-form compactly supported in the chart U_α, because

$$\operatorname{supp}(\rho_\alpha \omega) \subseteq \operatorname{supp} \rho_\alpha \cap \operatorname{supp} \omega \subseteq \operatorname{supp} \rho_\alpha \subseteq U_\alpha. \tag{8.41}$$

Hence, the integral $\int_{U_\alpha} \rho_\alpha \omega$ is defined, using (8.35). Therefore, we can define the inyehral of ω over M to be the finite sum

$$\int_M \omega := \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega. \tag{8.42}$$

(This is a finite sum, because $\rho_\alpha \omega$ is a nonzero form on U_α for only finitely many α) Now, for the integral (8.42) to be well-defined, we must show that $\int_M \omega$ is independent of the choices of oriented atlas and partition of unity. Let $\{(V_\beta, \psi_\beta)\}_{\beta \in B}$ be another oriented atlas specifying the same orientation as that of $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$. Suppose $\{\chi_\beta\}_{\beta \in B}$ be a partition of unity subordinate to the open cover $\{V_\beta\}_{\beta \in B}$. Then

$$\left\{\left(U_\alpha \cap V_\beta, \varphi_\alpha|_{U_\alpha \cap V_\beta}\right)\right\}_{\alpha,\beta} \text{ and } \left\{\left(U_\alpha \cap V_\beta, \psi_\beta|_{U_\alpha \cap V_\beta}\right)\right\}_{\alpha,\beta}$$

are two new atlases of M specifying the same orientation on M. Then one has

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha \sum_\beta \chi_\beta \omega, \tag{8.43}$$

since $\sum_\beta \chi_\beta = 1$. Now, the sum $\sum_\beta \chi_\beta \omega$ is, in fact, a finite sum, because $\chi_\beta \omega$ is a nonzero form on V_β for only finitely many β. Therefore, we can take the sum in front of the integral.

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\alpha \sum_\beta \int_{U_\alpha} \rho_\alpha \chi_\beta \omega. \tag{8.44}$$

Now,

$$\operatorname{supp}(\rho_\alpha \chi_\beta) \subseteq \operatorname{supp} \rho_\alpha \cap \operatorname{supp} \chi_\beta \subseteq U_\alpha \cap V_\beta. \tag{8.45}$$

Using (8.45), we can rewrite (8.44) as

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\alpha \sum_\beta \int_{U_\alpha \cap V_\beta} \rho_\alpha \chi_\beta \omega. \tag{8.46}$$

Similarly,

$$\sum_\beta \int_{V_\beta} \chi_\beta \omega = \sum_\beta \sum_\alpha \int_{V_\beta \cap U_\alpha} \chi_\beta \rho_\alpha \omega. \tag{8.47}$$

In (8.46) and (8.47), we can actually interchange the α and β sums, since they are finite sums. Therefore, we can conclude that

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\beta \int_{V_\beta} \chi_\beta \omega. \tag{8.48}$$

Therefore, the definition of the integral of a compactly supported smooth n-form on M given by (8.42) is independent of the choices of oriented atlas and the partition of unity subordinate to that atlas.

Proposition 8.6

Let ω be an n-form with compact support on an oriented manifold M of dimension n. If $-M$ is the same manifold with the opposite orientation, then

$$\int_{-M} \omega = -\int_M \omega. \tag{8.49}$$

Proof. By the definition an integral (8.35) and (8.10), it is enough to show that for every chart $(U, \varphi) = (U, x^1, x^2, \dots, x^n)$ in the oriented atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$, [we're dropping subscript α for notational clarity] and differential form $\tau \in \Omega^n_c(U)$, if $(U, \widetilde{\varphi}) = (U, -x^1, x^2, \dots, x^n)$ is the chart with the opposite orientation, then

$$\int_{\widetilde{\varphi}(U)} \left(\widetilde{\varphi}^{-1}\right)^* \tau = -\int \varphi(U) \left(\varphi^{-1}\right)^* \tau. \tag{8.50}$$

Let $r^1, \dots, r^n$ be the standard coordinates on $\mathbb{R}^n$. Then one has

$$x^i = r^i \circ \varphi \text{ and } r^i = x^i \circ \varphi^{-1}. \tag{8.51}$$

When one is dealing with the chart $(U, \widetilde{\varphi})$, (8.51) still remains true for $i = 2, 3, \dots, n$. Only the formula for $i = 1$, changes by a sign.

$$\begin{aligned} -x^1 &= r^1 \circ \widetilde{\varphi} \text{ and } r^1 = -x^1 \circ \widetilde{\varphi}^{-1}, \\ x^i &= r^i \circ \varphi \text{ and } r^i = x^i \circ \varphi^{-1}, \end{aligned} \tag{8.52}$$

for $i = 2, 3, \dots, n$. Now suppose,

$$\tau = f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

on U. Then

$$\begin{aligned} \left(\widetilde{\varphi}^{-1}\right)^* \tau &= \left(\widetilde{\varphi}^{-1}\right)^* \left(f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n\right) \\ &= \left(f \circ \widetilde{\varphi}^{-1}\right) \mathrm{d}\left(x^1 \circ \widetilde{\varphi}^{-1}\right) \wedge \cdots \wedge \mathrm{d}\left(x^n \circ \widetilde{\varphi}^{-1}\right) \\ &= -\left(f \circ \widetilde{\varphi}^{-1}\right) \mathrm{d}r^1 \wedge \cdots \wedge \mathrm{d}r^n. \end{aligned} \tag{8.53}$$

Similarly,

$$\left(\varphi^{-1}\right)^* \tau = \left(f \circ \varphi^{-1}\right) \mathrm{d}r^1 \wedge \cdots \wedge \mathrm{d}r^n. \tag{8.54}$$

Now take the map $\varphi \circ \widetilde{\varphi}^{-1} : \widetilde{\varphi}(U) \to \varphi(U)$. Take $(a^1, \dots, a^n) \in \widetilde{\varphi}(U)$. Let us compute $\left(r^i \circ \varphi \circ \widetilde{\varphi}^{-1}\right)(a^1, \dots, a^n)$. For $i = 1$,

$$\begin{aligned} \left(r^1 \circ \varphi \circ \widetilde{\varphi}^{-1}\right)\left(a^1, \dots, a^n\right) &= \left(x^1 \circ \widetilde{\varphi}^{-1}\right)\left(a^1, \dots, a^n\right) \\ &= -r^1\left(a^1, \dots, a^n\right) = -a^1. \end{aligned} \tag{8.55}$$

For $i = 2, 3, \dots, n$,

$$\begin{aligned} \left(r^i \circ \varphi \circ \widetilde{\varphi}^{-1}\right)\left(a^1, \dots, a^n\right) &= \left(x^i \circ \widetilde{\varphi}^{-1}\right)\left(a^1, \dots, a^n\right) \\ &= r^i\left(a^1, \dots, a^n\right) = a^i. \end{aligned} \tag{8.56}$$

Therefore,

$$\left(\varphi \circ \widetilde{\varphi}^{-1}\right)\left(a^1, a^2, \dots, a^n\right) = \left(-a^1, a^2, \dots, a^n\right). \tag{8.57}$$

So, the Jacobian matrix of $\varphi \circ \widetilde{\varphi}^{-1}$ will be a diagonal matrix, with entries $-1, 1, \dots, 1$. Hence,

$$\left|\det J\left(\varphi \circ \widetilde{\varphi}^{-1}\right)\right| = |-1| = 1. \tag{8.58}$$

Therefore,

$$\begin{aligned} \int_{\widetilde{\varphi}(U)} \left(\widetilde{\varphi}^{-1}\right)^* \tau &= -\int_{\widetilde{\varphi}(U)} \left(f \circ \widetilde{\varphi}^{-1}\right) \mathrm{d}r^1 \cdots \mathrm{d}r^n \\ &= -\int_{\widetilde{\varphi}(U)} \left(f \circ \varphi^{-1}\right) \circ \left(\varphi \circ \widetilde{\varphi}^{-1}\right) \left|\det J\left(\varphi \circ \widetilde{\varphi}^{-1}\right)\right| \mathrm{d}r^1 \cdots \mathrm{d}r^n \\ &= -\int_{\varphi(U)} \left(f \circ \varphi^{-1}\right) \mathrm{d}r^1 \cdots \mathrm{d}r^n \\ &= -\int \varphi(U) \left(\varphi^{-1}\right)^* \tau. \end{aligned}$$

Therefore, (8.50) holds. By the linearity of integration, this proves (8.49). ■

In practical computation, the definition of integral of an n-form over an oriented n-manifold using partition of unity is not very useful. To calculate explicitly integrals over an oriented n-manifold M, it's best to consider integrals over a parametrized set.

> **Definition 8.6** (Paramtrized set)**.** A **parametrized set** in an oriented n-manifold M is a subset $A \subseteq M$ together with a C^∞ map $F : D \to M$ from a compact domain of integration $D \subset \mathbb{R}^n$ to M such that $A = F(D)$ and F restricts to an orientation preserving diffeomorphism from $\operatorname{int}(D)$ to $F(\operatorname{int} D)$. Note that by smooth invariance of domain for manifolds (Remark 7.2), $F(\operatorname{int} D)$ is an open subset of M. The C^∞ map $F : D \to A$ is called a **parametrization** of A.

If $A \subseteq M$ is a parametrized set with parametrization $F : D \to A$ and ω is a C^∞ n-form on M, not necessarily with compact support, then we define $\int_A \omega$ to be $\int_D F^*\omega$.

$$\int_A \omega := \int_D F^*\omega. \tag{8.59}$$

One can show that this definition is parametrization independent. Indeed, if there is a C^∞ map $\widetilde{F} : \widetilde{D} \to M$ from a compact domain of integration $\widetilde{D} \subset \mathbb{R}^n$ to M such that $A = \widetilde{F}(\widetilde{D})$ and $\widetilde{F}$ restricts to an orientation preserving diffeomorphism from $\operatorname{int}\left(\widetilde{D}\right)$ to $\widetilde{F}\left(\operatorname{int} \widetilde{D}\right)$, them

$$\int_{\widetilde{D}} \widetilde{F}^*\omega = \int_D F^*\omega.$$

It can be seen by showing that there is a smooth map $G : \widetilde{D} \to D$ which restricts to an orientation preserving diffeomorphism from $\operatorname{int}\left(\widetilde{D}\right)$ to $\operatorname{int}(D)$, such that

$$F \circ G = \widetilde{F}. \tag{8.60}$$

Then by Theorem 5.7,

$$\widetilde{F}^* = G^* \circ F^*. \tag{8.61}$$

Then by the definition of integration over parametrized set,

$$\int_D F^*\omega = \int_{\widetilde{D}} G^*\left(F^*\omega\right) = \int_{\widetilde{D}} \widetilde{F}^*\omega. \tag{8.62}$$

It's important to note that $\int_D F^*\omega$ and $\int_{\widetilde{D}} \widetilde{F}^*\omega$ are equal and do not differ by a sign is because $G : \widetilde{D} \to D$ restricts to an orientation preserving diffecomorphism from $\operatorname{int}(\widetilde{D})$ to $G(\operatorname{int}(\widetilde{D})) = \operatorname{int} D$.

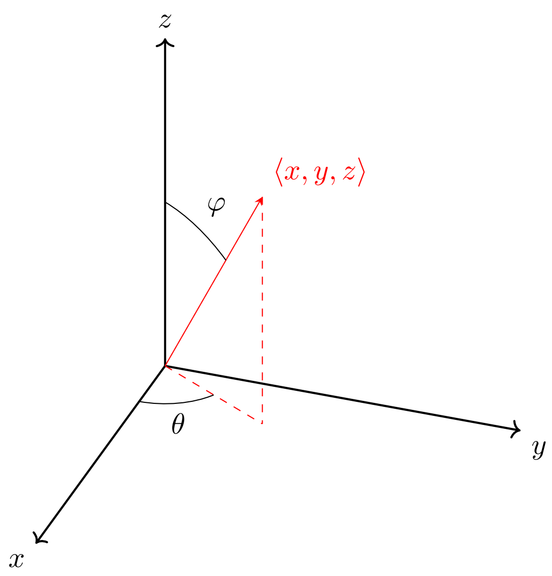
In case the parametrized set $A = F(D) \subseteq M$ is a manifold, then $\int_A \omega = \int_{D\subset\mathbb{R}^n} F^*\omega$ and the definition (8.35) $\int_U \omega = \int_{\phi(U)\subset\mathbb{R}^n} \left(\phi^{-1}\right)^* \omega$ coincide by looking at the smooth maps $F : D \subset \mathbb{R}^n \to F(D) = A$ and $\phi^{-1} : \phi(U) \subset \mathbb{R}^n \to U$. In the former case $D \subset \mathbb{R}^n$ is taken to be compact so that we don't want ω to be compactly supported in this case while in the latter case we have $\phi(U) \subset \mathbb{R}^n$ to be open or in other words U to be open so that in this case we required ω to be compactly supported inside the open subset U of M.

The theory of integration using parametrizing sets is computationally handy. We refer the interested reader to the treatment in *Analysis on Manifolds* by James Munkres (there again Munkres used parametrized open sets in contrast to the compact set $A = F(D)$ we used and hence Munkres needed the n-form ω to be compactly supported inside the parametrized open set). We try to content ourselves with an example.

Example 8.1 (Integral over a sphere)**.** Given unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ along the x-axis, y-axis and z-axis, respectively, the vector $\langle x, y, z\rangle$ from the origin $(0, 0, 0)$ to the point (x, y, z) is nothing but $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then we take

$$r = \sqrt{x^2 + y^2 + z^2} = \|\langle x, y, z\rangle\| = \|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\|. \tag{8.63}$$

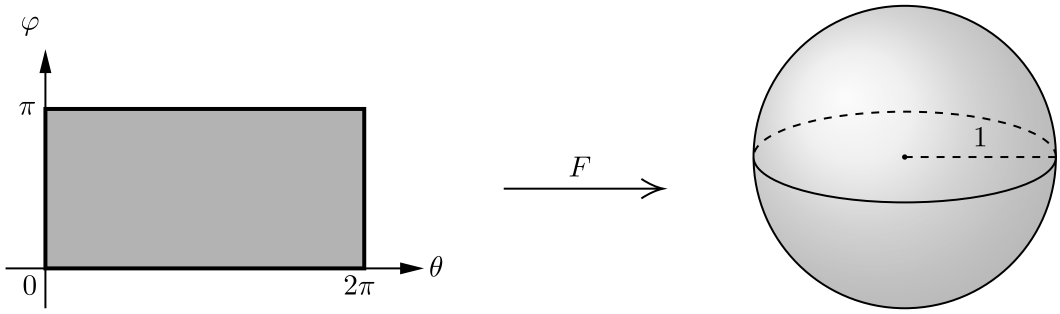
The set of all points in $\mathbb{R}^3$ obeying $\|\langle x, y, z\rangle\| = r$ for a given positive real r is the sphere of radius r centered at the origin $(0, 0, 0)$. Let us denote by φ the angle $\langle x, y, z\rangle$ makes with the positive z-axis, and denote by θ the angle that the vector $\langle x, y\rangle$ makes with the positive x-axis.



Let S^2 be the unit sphere in $\mathbb{R}^3$ $(r = 1)$. Let ω be the 2-form on S^2 given by

$$\omega = \begin{cases} \frac{\mathrm{d}y \wedge \mathrm{d}z}{x} & \text{if } x \neq 0, \\ \frac{\mathrm{d}z \wedge \mathrm{d}x}{y} & \text{if } y \neq 0, \\ \frac{\mathrm{d}x \wedge \mathrm{d}y}{z} & \text{if } z \neq 0. \end{cases} \tag{8.64}$$

We want to calculate $\int_{S^2} \omega$.



In this problem, the compact domain $D \subset \mathbb{R}^2$ of integration is given by

$$D = \left\{ (\theta, \varphi) \in \mathbb{R}^2 \mid 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi \right\}. \tag{8.65}$$

Here, $S^2 \subset \mathbb{R}^3$ is the parametrized set which happens to be a 2-dimensional manifold, and the parametrization is the smooth map $F : D \to S^2$ given by

$$F(\theta, \varphi) = (\sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi) = (x, y, z) \in S^2 \subset \mathbb{R}^3. \tag{8.66}$$

Here x, y, z are 3 coordinate functions (0-forms) on S^2, so that the pullbacks F^*x, F^*y and F^*z under the smooth map F given by are expected to be functions in D. Indeed,

$$\begin{aligned} F^*x &= x \circ F = \sin\varphi \cos\theta, \\ F^*y &= y \circ F = \sin\varphi \sin\theta, \\ F^*z &= z \circ F = \cos\varphi. \end{aligned} \tag{8.67}$$

Thus, we have

$$F^*(\mathrm{d}x) = \mathrm{d}(F^*x) = \mathrm{d}(\sin\varphi \cos\theta) = \cos\varphi \cos\theta \, \mathrm{d}\varphi - \sin\theta \sin\varphi \, \mathrm{d}\theta. \tag{8.68}$$

$$F^*(\mathrm{d}y) = \mathrm{d}(F^*y) = \mathrm{d}(\sin\varphi \sin\theta) = \cos\varphi \sin\theta \, \mathrm{d}\varphi + \sin\varphi \cos\theta \, \mathrm{d}\theta. \tag{8.69}$$

$$F^*(\mathrm{d}z) = \mathrm{d}(F^*z) = \mathrm{d}(\cos\theta) = -\sin\theta \, \mathrm{d}\theta. \tag{8.70}$$

Now, for $x \neq 0$,

$$F^*\omega = \frac{F^*\mathrm{d}y \wedge F^*\mathrm{d}z}{F^*x} = \frac{(\cos\varphi\sin\theta\,\mathrm{d}\varphi + \sin\varphi\cos\theta\,\mathrm{d}\theta)\wedge(-\sin\theta\,\mathrm{d}\theta)}{\sin\varphi\cos\theta}$$
$$= \frac{\sin^2\theta\cos\theta\,\mathrm{d}\varphi\wedge\mathrm{d}\theta}{\sin\varphi\cos\theta} = \sin\theta\,\mathrm{d}\varphi\wedge\mathrm{d}\theta.$$

For $y \neq 0$,

$$F^*\omega = \frac{F^*\mathrm{d}z \wedge F^*\mathrm{d}x}{F^*y} = \frac{(-\sin\theta\,\mathrm{d}\theta)\wedge(\cos\varphi\cos\theta\,\mathrm{d}\varphi - \sin\theta\sin\varphi\,\mathrm{d}\theta)}{\sin\varphi\sin\theta}$$
$$= \frac{\sin^2\varphi\sin\theta\,\mathrm{d}\varphi\wedge\mathrm{d}\theta}{\sin\varphi\sin\theta} = \sin\theta\,\mathrm{d}\varphi\wedge\mathrm{d}\theta.$$

For $z \neq 0$,

$$F^*\omega = \frac{F^*\mathrm{d}x \wedge F^*\mathrm{d}y}{F^*z} = \frac{(\cos\varphi\cos\theta\,\mathrm{d}\varphi - \sin\theta\sin\varphi\,\mathrm{d}\theta)\wedge(\cos\varphi\sin\theta\,\mathrm{d}\varphi + \sin\varphi\cos\theta\,\mathrm{d}\theta)}{\cos\varphi}$$
$$= \frac{(\sin\varphi\cos\varphi\cos^2\theta + \sin\varphi\cos\varphi\sin^2\theta)\,\mathrm{d}\varphi\wedge\mathrm{d}\theta}{\cos\varphi}$$
$$= \frac{\sin\varphi\cos\varphi\,\mathrm{d}\varphi\wedge\mathrm{d}\theta}{\cos\varphi} = \sin\theta\,\mathrm{d}\varphi\wedge\mathrm{d}\theta.$$

Therefore,

$$F^*\omega = \sin\theta\,\mathrm{d}\varphi\wedge\mathrm{d}\theta, \tag{8.71}$$

everywhere. Now using the definition of integral over a parametrized set,

$$\int_{S^2}\omega = \int_D F^*\omega = \int_0^{2\pi}\int_0^{\pi}\sin\varphi\,\mathrm{d}\varphi\ \mathrm{d}\theta$$
$$= 2\pi\left[-\cos\varphi\right]\Big|_{\varphi=0}^{\varphi=\pi} = 4\pi, \tag{8.72}$$

which is the surface area of the unit sphere S^1.

§8.4 Stokes' Theorem

Theorem 8.7 (Stokes' Theorem)
Let M be an oriented smooth n-manifold with boundary, and let ω be a compactly supported smooth $(n-1)$-form on M. Then

$$\int_M \mathrm{d}\omega = \int_{\partial M}\omega. \tag{8.73}$$

∂M is a smooth $(n-1)$-manifold without any boundary as we've sen in the previous lectures. There is an indued orientation on ∂M. The $(n-1)$ form ω appearing on the right side of (8.73) is to be interpreted as $\iota_{\partial M}^*\omega$, where

$$\iota_{\partial M} : \partial M \to M$$

is the canonical inclusion of the boundary ∂M into the n-manifold M. If $\partial M = \varnothing$, then the right side is to be interpreted as zero. When M is 1-dimensional, the right hand integral is just a finite sum in the following sense:

Integration over a zero-dimensional manifold

A compact oriented manifold of dimension 0 is a finite collection of points, each point oriented by +1 or −1 We write this fact as

$$M=\sum_{i=1}^{r} p_i-\sum_{j=1}^{s} q_j, \tag{8.74}$$

which means that each element of the collection $\{p_1, p_2, \ldots, p_r\}$ has orientation +1 while each element of the collection $\{q_1, q_2, \ldots, q_s\}$ has orientation −1. The object that is to be integrated over this oriented 0-dimensions manifold M is a 0-form $f : M \to \mathbb{R}$. The integral of this 0-form is defined to be the sum

$$\int_M f:=\sum_{i=1}^{r} f\left(p_i\right)-\sum_{j=1}^{s} f\left(q_j\right) \tag{8.75}$$

We need the following result to prove Stokes' Theorem.

Lemma 8.8

Suppose M is a smooth manifold with or without boundary and $S \subseteq M$ is an immersed submanifold with or without boundary. Let $\iota : S \to M$ be the relevant inclusion. Then

$$\mathrm{d}\left(\left.f\right|_S\right)=\iota^*(\mathrm{d} f).$$

Furthermore, the pullback of $\mathrm{d}f$ to S is zero if and only if f is constant on each component of S.

Proof. $\left.f\right|_S = f \circ \iota : S \to \mathbb{R}$. Therefore,

$$\mathrm{d}\left(\left.f\right|_S\right)=\mathrm{d}(f \circ \iota)=\mathrm{d}\left(\iota^* f\right)=\iota^*(\mathrm{d} f). \tag{8.76}$$

The pullback of $\mathrm{d}f$ to S is $\iota^*(\mathrm{d}f)$, which is equal to $\mathrm{d}\left(\left.f\right|_S\right)$ as we have just proved. This will be zero if and only if $\mathrm{d}\left(\left.f\right|_S\right) = 0$.

Let $g = \left.f\right|_S$. We need to show that $\mathrm{d}g = 0$ if and only if g is constant on each component of S. Suppose g is constant on each component of S. Let $(U, x^1, \ldots, x^m)$ be a (connected) chart on S. Then on U,

$$\mathrm{d} g=\sum_{i=1}^{m} \frac{\partial g}{\partial x^i} \mathrm{~d} x^i=0. \tag{8.77}$$

So $\mathrm{d}g = 0$ on all of S, since U was an arbitrary connected coordinate open set.

Conversely, suppose $\mathrm{d}g = 0$. Let $(U, \varphi) \equiv (U, x^1, \ldots, x^m)$ be a (connected) chart on S. Then on U,

$$0=\mathrm{d} g=\sum_{i=1}^{m} \frac{\partial g}{\partial x^i} \mathrm{~d} x^i. \tag{8.78}$$

This gives us that

$$\frac{\partial g}{\partial x^i}=0, \tag{8.79}$$

for each $i = 1, \ldots, m$. Then we get

$$\frac{\partial}{\partial r^i}\left(g \circ \varphi^{-1}\right)=0 \tag{8.80}$$

on $\varphi(U)$. Since $\varphi(U)$ is a connected open subset of $\mathbb{R}^m$, we can conclude that $g \circ \varphi^{-1}$ is constant on $\varphi(U)$. As a result, g is constant on U.

Let V be another connected coordinate open subset of S, belonging to the same connected component of S as U, such that $U \cap V \neq \varnothing$ (if there doesn't exist such V, then U is a connected component of S). Using the same argument as above, we conclude that g is constant on V as well. Since the constants must agree on $U \cap V$, we must have g to be constant on $U \cup V$. Continuing like this, we conclude that g is constant on the connected component that contains U. Therefore, g is constant on each connected compoent of S. ■

Proof of Stokes' Theorem. We begin with the special case when M is the upper half-space $\mathbb{H}^n$ itself, for $n > 1$. Since ω is a compactly supported $(n-1)$-form in $\mathbb{H}^n$, there is a real number $K > 0$ such that $\operatorname{supp}\omega$ is contained in the rectangle

$$R = [-K, K] \times \cdots \times [-K, K] \times [0, K] .$$

We can write ω using the standard coordinates of $\mathbb{H}^n$ as

$$\omega = \sum_{i=1}^{n} \omega_i \, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n, \tag{8.81}$$

where, as before, the caret stands for omission. Hence we have

$$\begin{aligned}
\mathrm{d}\omega &= \sum_{i=1}^{n} \mathrm{d}\omega_i \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \omega_i}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \\
&= \sum_{i=1}^{n} \frac{\partial \omega_i}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n.
\end{aligned} \tag{8.82}$$

Here in the j-sum, the $i \neq j$ terms will vanish since $\mathrm{d}x^k \wedge \mathrm{d}x^j = 0$ for $k \neq j$. In order to reinstate the first $\mathrm{d}x^i$ in its original position (where the caret occurs), it has to be pushed through the wedge product $(i-1)$-times and hence (8.82) reduces to

$$\mathrm{d}\omega = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^i \wedge \cdots \wedge \mathrm{d}x^n. \tag{8.83}$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{H}^n} \mathrm{d}\omega &= \sum_{i=1}^{n} (-1)^{i-1} \int_R \frac{\partial \omega_i}{\partial x^i} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\
&= \sum_{i=1}^{n} (-1)^{i-1} \int_0^K \int_{-K}^K \cdots \int_{-K}^K \frac{\partial \omega_i}{\partial x^i} \, \mathrm{d}x^1 \cdots \mathrm{d}x^n.
\end{aligned} \tag{8.84}$$

The contribution of the terms with $i \neq n$ in this sum are:

$$\begin{aligned}
&\int_0^K \int_{-K}^K \cdots \int_{-K}^K \frac{\partial \omega_i}{\partial x^i} \, \mathrm{d}x^1 \cdots \mathrm{d}x^n \\
&= \int_0^K \int_{-K}^K \cdots \int_{-K}^K \left[\omega_i(x)\right]_{x^i=-K}^{x^i=K} \, \mathrm{d}x^1 \cdots \widehat{\mathrm{d}x^i} \cdots \mathrm{d}x^n \\
&= 0,
\end{aligned} \tag{8.85}$$

since we can choose K large enough so that $\omega = 0$ when $x^i = \pm K$. As a result,

$$\begin{aligned}
\int_{\mathbb{H}^n} \mathrm{d}\omega &= (-1)^{n-1} \int_0^K \int_{-K}^K \cdots \int_{-K}^K \frac{\partial \omega_n}{\partial x^n} \, \mathrm{d}x^1 \cdots \mathrm{d}x^n \\
&= (-1)^{n-1} \int_{-K}^K \cdots \int_{-K}^K \left[\omega_n(x)\right]_{x^n=0}^{x^n=K} \, \mathrm{d}x^1 \cdots \mathrm{d}x^{n-1} \\
&= (-1)^n \int_{-K}^K \cdots \int_{-K}^K \left[\omega_n\left(x^1, \ldots, x^{n-1}, 0\right)\right]_{x^n=0}^{x^n=K} \, \mathrm{d}x^1 \cdots \mathrm{d}x^{n-1},
\end{aligned} \tag{8.86}$$

again by choosing K large enough so that $\omega = 0$ when $x^n = K$.

Let us now compute $\int_{\partial \mathbb{H}^n} \omega$.

$$\int_{\partial \mathbb{H}^n} \omega = \sum_{i=1}^{n} \int_{R \cap \partial \mathbb{H}^n} \omega_i\left(x^1, \ldots, x^{n-1}, 0\right) \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n. \tag{8.87}$$

On $\partial\mathbb{H}^n$, $x^n = 0$, i.e. x^n is constant on $\partial\mathbb{H}^n$. Therefore, by Lemma 8.8,

$$\iota^* \left(\mathrm{d}x^n\right) = \mathrm{d}\left(x^n\big|_{\partial\mathbb{H}^n}\right) = 0, \tag{8.88}$$

where $\iota : \partial\mathbb{H}^n \to \mathbb{H}^n$ is the inclusion map. Hence, in (8.87), for the sum over all i, only the term corresponding to $i = n$ survives with $\mathrm{d}x^n$ being omitted in that term. Therefore,

$$\int_{\partial\mathbb{H}^n} \omega = \int_{R\cap\partial\mathbb{H}^n} \omega_n \left(x^1, \ldots, x^{n-1}, 0\right) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{n-1}. \tag{8.89}$$

By Example 7.4, the induced orientation on $\partial\mathbb{H}^n$ is given by the $(n-1)$-form $(-1)^n \,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{n-1}$. Using this orientation, (8.89) reads

$$\int_{\partial\mathbb{H}^n} \omega = (-1)^n \int_{-K}^{K} \cdots \int_{-K}^{K} \omega_n \left(x^1, \ldots, x^{n-1}, 0\right) \,\mathrm{d}x^1 \cdots \mathrm{d}x^{n-1}. \tag{8.90}$$

Comparing (8.86) and (8.90), we get

$$\int_{\mathbb{H}^n} \mathrm{d}\omega = \int_{\partial\mathbb{H}^n} \omega. \tag{8.91}$$

Now we consider the special case when $M = \mathbb{H}^1$ or $\mathbb{L}^1$. Let f be a smooth compactly supported 0-form on $\mathbb{H}^1$. Since f is compactly supported, there exists some $K > 0$ such that $\operatorname{supp} f \subseteq [0, K]$. Now,

$$\mathrm{d}f = \frac{\partial f}{\partial x} \,\mathrm{d}x. \tag{8.92}$$

So we have

$$\int_{\mathbb{H}^1} \mathrm{d}f = \int_{\mathbb{H}^1} \frac{\partial f}{\partial x} \wedge \mathrm{d}x = \int_0^K \frac{\partial f}{\partial x} \,\mathrm{d}x = f(K) - f(0) = -f(0), \tag{8.93}$$

since we can choose K large enough so that $f(K) = 0$. Now, the canonical orientation form $\mathrm{d}x$ on $\mathbb{H}^1$ induces an orientation on $\partial\mathbb{H}^1 = \{0\}$, which is -1 (by Example 7.4). Therefore, by the definition of integral of 0-forms on 0-dimensional manifold,

$$\int_{\partial\mathbb{H}^1} f = -f(0). \tag{8.94}$$

Comparing (8.93) and (8.94), we get

$$\int_{\mathbb{H}^1} \mathrm{d}f = \int_{\partial\mathbb{H}^1} f. \tag{8.95}$$

Now consider $M = \mathbb{L}^1$. The canonical orientation form on $\mathbb{L}^1$ is $\mathrm{d}x$. A smooth outward pointing vector field on $\mathbb{L}^1$ is $\frac{\partial}{\partial x}$. Therefore, the canoncial boundary orientation on $\partial\mathbb{L}^1 = \{0\}$ is given by the contraction

$$\iota_{\frac{\partial}{\partial x}} (\mathrm{d}x), \tag{8.96}$$

by Proposition 7.8. Since $\iota_{\mathbf{v}}\beta = \beta(\mathbf{v}) \in \mathbb{R}$, we have

$$\iota_{\frac{\partial}{\partial x}} (\mathrm{d}x) = \mathrm{d}x\left(\frac{\partial}{\partial x}\right) = 1. \tag{8.97}$$

Since f is compactly supported, there exists some $K > 0$ such that $\operatorname{supp} f \subseteq [-K, 0]$. Now,

$$\mathrm{d}f = \frac{\partial f}{\partial x} \,\mathrm{d}x. \tag{8.98}$$

So we have

$$\int_{\mathbb{L}^1} \mathrm{d}f = \int_{\mathbb{L}^1} \frac{\partial f}{\partial x} \wedge \mathrm{d}x = \int_{-K}^0 \frac{\partial f}{\partial x} \,\mathrm{d}x = f(0) - f(-K) = f(0), \tag{8.99}$$

since we can choose K large enough so that $f(-K) = 0$. Since the orientation on $\partial\mathbb{L}^1 = \{0\}$ is $+1$, by the definition of integral of 0-forms on 0-dimensional manifold,

$$\int_{\partial\mathbb{L}^1} f = f(0). \tag{8.100}$$

Comparing (8.99) and (8.100), we get

$$\int_{\mathbb{L}^1} \mathrm{d}f = \int_{\partial \mathbb{L}^1} f. \tag{8.101}$$

Next we consider the special case $M = \mathbb{R}^n$. Since ω is a compactly supported $(n-1)$-form on $\mathbb{R}^n$, there exists some $K > 0$ such that $\operatorname{supp}\omega \subseteq R = [-K, K]^n$. Then we write

$$\omega = \sum_{i=1}^{n} \omega_i \,\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n. \tag{8.102}$$

Then

$$\begin{aligned}
\mathrm{d}\omega &= \sum_{i=1}^{n} \frac{\partial \omega_i}{\partial x^i} \,\mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \\
&= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} \,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^i \wedge \cdots \wedge \mathrm{d}x^n.
\end{aligned} \tag{8.103}$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathrm{d}\omega &= \sum_{i=1}^{n} (-1)^{i-1} \int_R \frac{\partial \omega_i}{\partial x^i} \,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\
&= \sum_{i=1}^{n} (-1)^{i-1} \int_{-K}^{K} \cdots \int_{-K}^{K} \frac{\partial \omega_i}{\partial x^i} \,\mathrm{d}x^1 \cdots \mathrm{d}x^n.
\end{aligned} \tag{8.104}$$

Let us now compute the integrals:

$$\begin{aligned}
&\int_{-K}^{K} \cdots \int_{-K}^{K} \frac{\partial \omega_i}{\partial x^i} \,\mathrm{d}x^1 \cdots \mathrm{d}x^n \\
&= \int_{-K}^{K} \cdots \int_{-K}^{K} \left[\omega_i\left(x\right)\right]_{x^i=-K}^{x^i=K} \,\mathrm{d}x^1 \cdots \widehat{\mathrm{d}x^i} \cdots \mathrm{d}x^n \\
&= 0,
\end{aligned} \tag{8.105}$$

since we can choose K large enough so that $\omega = 0$ when $x^i = \pm K$. Therefore,

$$\int_{\mathbb{R}^n} \mathrm{d}\omega = 0. \tag{8.106}$$

Since $\mathbb{R}^n$ has empty boundary, i.e. $\partial \mathbb{R}^n = \varnothing$,

$$\int_{\partial \mathbb{R}^n} \omega = \int_{\varnothing} \omega = 0. \tag{8.107}$$

Hence,

$$\int_{\mathbb{R}^n} \mathrm{d}\omega = \int_{\partial \mathbb{R}^n} \omega. \tag{8.108}$$

So we have proved Stokes' Theorem for the special cases $M = \mathbb{H}^n, \mathbb{L}^1, \mathbb{R}^n$. Now let M be an arbitrary smooth manifold with boundary ∂M. Choose an atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ for M in which each U_α is diffeomorphic to either $\mathbb{R}^n$ or $\mathbb{H}^n$ (or $\mathbb{L}^1$ in dimension 1) via an orientation preserving diffeomorphism. This is possible since any open disk is diffeomorphic to $\mathbb{R}^n$ and any half-disk containing its boundary diameter is diffeomorphic to $\mathbb{H}^n$ (or $\mathbb{L}^1$ in dimension 1). For instance, the open ball

$$B = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| < 1\} \tag{8.109}$$

is diffeomorphic to $\mathbb{R}^n$ via the map $F : B \to \mathbb{R}^n$ defined as

$$F\left(\mathbf{x}\right) = \frac{1}{\sqrt{1 - \|\mathbf{x}\|^2}} \mathbf{x}. \tag{8.110}$$

If this map is not orientation preserving, we just take the map $\widetilde{F} = (-F^1, F^2, \dots, F^n)$. Then the first row of $\left[\frac{\partial \widetilde{F}^i}{\partial x^j}\right]_{i,j=1}^n$ is the negative of the first row of $\left[\frac{\partial F^i}{\partial x^j}\right]_{i,j=1}^n$, and all the other rows stay the same. Therefore,

$$\det \left[\frac{\partial \widetilde{F}^i}{\partial x^j}\right]_{i,j=1}^n = -\det \left[\frac{\partial F^i}{\partial x^j}\right]_{i,j=1}^n . \tag{8.111}$$

So $\widetilde{F}$ is orientation preserving. In the same way, the half-disk containing its boundary diameter $B \cap \mathbb{H}^n$ is diffeomorphic to $\mathbb{H}^n$ (or $\mathbb{L}^1$) via an orientation preserving diffeomorphism.

Let $\{\rho_\alpha\}_\alpha$ be a C^∞ partition of unity subordinate to $\{U_\alpha\}$. From (8.41), $\operatorname{supp} \rho_\alpha \omega \subseteq U_\alpha$. Furthermore, by (8.39), $\operatorname{supp}(\rho_\alpha \omega) \subseteq \operatorname{supp} \omega$, i.e. $\operatorname{supp}(\rho_\alpha \omega)$ is a closed subset of a compact set $\operatorname{supp} \omega$. Hence, $\operatorname{supp}(\rho_\alpha \omega)$ is compact. In other words, $\rho_\alpha \omega$ has compact support in U_α. Furthermore, $\omega = \sum_\alpha \rho_\alpha \omega$ is, in fact, a finite sum, as proven earlier.

Since Stokes' Theorem holds for $\mathbb{R}^n$ and $\mathbb{H}^n$ (and $\mathbb{L}^1$), it holds for all the charts U_α in our atlas, with each U_α being diffeomorphic to $\mathbb{R}^n$ or $\mathbb{H}^n$ (or $\mathbb{L}^1$) and preserving orientation. Furthermore, $\partial M \cap U_\alpha = \partial U_\alpha$. Therefore,

$$\begin{aligned}
\int_{\partial M} \omega &= \int_{\partial M} \sum_\alpha \rho_\alpha \omega \\
&= \sum_\alpha \int_{\partial M} \rho_\alpha \omega && [\textstyle\sum_\alpha \rho_\alpha \omega \text{ is a finite sum}] \\
&= \sum_\alpha \int_{\partial U_\alpha} \rho_\alpha \omega && [\operatorname{supp}(\rho_\alpha \omega) \subseteq U_\alpha] \\
&= \sum_\alpha \int_{U_\alpha} \mathrm{d}(\rho_\alpha \omega) && [\text{Stokes' Theorem for } U_\alpha] \\
&= \sum_\alpha \int_{M} \mathrm{d}(\rho_\alpha \omega) && [\operatorname{supp} \mathrm{d}(\rho_\alpha \omega) \subseteq \operatorname{supp}(\rho_\alpha \omega) \subseteq U_\alpha] \\
&= \int_M \mathrm{d}\left(\sum_\alpha \rho_\alpha \omega\right) && [\textstyle\sum_\alpha \rho_\alpha \omega \text{ is a finite sum}] \\
&= \int_M \mathrm{d}\omega.
\end{aligned}$$

Therefore, for any manifold M,

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega. \tag{8.112}$$

■

9 de Rham Cohomology

§9.1 Definitions

A differential form ω on a manifold M is said to be **closed** if $\mathrm{d}\omega = 0$, and **exact** if $\omega = \mathrm{d}\tau$ for some form τ of one degree less. Let us denote by $Z^k(M)$ the vector space of all closed k-forms on M, and by $B^k(M)$ the vector space of all exact k-forms on M. Since the exterior derivative operator d satisfies $\mathrm{d}^2 = 0$, $B^K(M)$ is a subspace of $Z^k(M)$. The quotient vector space

$$H^k(M) = \frac{Z^k(M)}{B^k(M)}$$

measures the extent to which closed k-forms fail to be exact, and is called the **de Rham cohomology** of M in degree k.

The quotient vector space construction introduces an equivalence relation on $Z^k(M)$: two closed k-forms on M are said to be equivalent if they differ by an exact k-form. In other words, $\omega' \sim \omega$ in $Z^k(M)$ if and only if $\omega' - \omega \in B^k(M)$, i.e.

$$\omega' - \omega = \mathrm{d}\tau, \tag{9.1}$$

for some $\tau \in \Omega^{k-1}(M)$. The equivalence class of a closed form ω is called its **cohomology class**, and is denoted by $[\omega]$. When (9.1) holds, we say that ω and ω' are **cohomologous**.

Proposition 9.1

If the manifold M has r connected components, then its de Rham cohomology in degree 0 is $H^0(M) = \mathbb{R}^r$. An element of $H^0(M)$ is specified by an ordered r tuple of real numbers, each real number representing a constant function on a connected component of M.

Proof. Since there are no nonzero exact 0-forms (smooth functions),

$$H^0(M) = Z^0(M) = \{\text{closed 0-forms }\}. \tag{9.2}$$

suppose f is a closed 0-form on M, i.e. $f \in C^\infty(M)$ such that $\mathrm{d}f = 0$. On any chart $(U, x^1, \ldots, x^4)$,

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i. \tag{9.3}$$

Thus $\mathrm{d}f = 0$ if and only if all the partial derivatives $\frac{\partial f}{\partial x^i}$ vanish identically on U. This implies that f is locally constant on U. It means that closed 0-forms on M are the locally constant functions on M. Such a function has to be constant on each connected component of M. If M has r connected components, then such a locally constant function is represented by an ordered r tuple of real numbers. In other words, $Z^0(M) = \mathbb{R}^r$. ■

Proposition 9.2

On a manifold M of dimension n, the de Rham cohomology $H^k(M)$ vanishes for $k > n$.

Proof. Given $p \in M$, the tangent space T_pM is a vector space of dimension n. If ω is a k-form on M, then $\omega_p \in A_k(T_pM)$, a k-covector or an alternating k-linear function on the vector space T_pM. By Corollary 1.17, if $k > n$, then $A_k(T_pM) = 0$. Hence, for $k > n$, the only k-form is just the zero form. ■

Example 9.1 (de Rham cohomology of the real line $\mathbb{R}$)**.** Since the real line $\mathbb{R}$ is connected, by Proposition 9.1, $H^0(\mathbb{R}) = \mathbb{R}$. By Proposition 9.2, $H^k(\mathbb{R}) = 0$ for $k \geq 2$. It remains to find $H^1(\mathbb{R})$.

There are no nonzero 2-forms on $\mathbb{R}$. Any 1-form on $\mathbb{R}$ can be expressed as $\omega = f\,\mathrm{d}x$, where $f \in C^\infty(\mathbb{R})$. One then has

$$\mathrm{d}\omega = \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x \wedge \mathrm{d}x = 0. \tag{9.4}$$

In other words, every 1-form on $\mathbb{R}$ is closed. Now, a 1-form $\omega = f\,\mathrm{d}x$ on $\mathbb{R}$ is exact if and only if there is a C^∞ function g on $\mathbb{R}$ such that

$$\omega = f\,\mathrm{d}x = \mathrm{d}g = g'\,\mathrm{d}x, \tag{9.5}$$

where g' is the calculus derivative of g with respect to x. Such a function g is simply an anti derivative of f. By fundamental theorem of calculus, one indeed finds from the following

$$g(x) = \int_0^x f(t)\,\mathrm{d}t, \tag{9.6}$$

that $g'(x) = f(x)$. Hence, every 1-form $\omega = f\,\mathrm{d}x$ on $\mathbb{R}$ is exact. Hence, $Z^1(\mathbb{R}) = B'(\mathbb{R})$, resulting in

$$H^1(\mathbb{R}) = \frac{Z'(\mathbb{R})}{B^1(\mathbb{R})} = 0. \tag{9.7}$$

One, therefore, has

$$H^k(\mathbb{R}) = \begin{cases} \mathbb{R} & \text{for } k = 0, \\ 0 & \text{for } k \geq 1. \end{cases} \tag{9.8}$$

Example 9.2 (de Rham cohomology of a circle)**.** Let S^1 be the circle in the x-y place. Since S^1 is connected, by Proposition 9.1, $H^0(S^1) = \mathbb{R}$. Since S^1 is a one-dimensional manifold, by Proposition 9.2, $H^k(S^1) = 0$ for $k \geq 2$. Now we need to compute $H^1(S^1)$.

Let us first consider the map $h : \mathbb{R} \to S^1$ defined by

$$h(t) = (\cos t, \sin t). \tag{9.9}$$

Let $i : [0, 2\pi] \to \mathbb{R}$ be the inclusion map. Restriction of the domain of $h : \mathbb{R} \to S^1$ is obtained by the following composition $F = h \circ i : [0, 2\pi] \to S^1$, which is a parametrization of the circle. In Example 5.2, we found a nowhere vanishing 1-form $\omega = -y\,\mathrm{d}x + x\,\mathrm{d}y$ on S^1. Now,

$$\begin{aligned} h^*(-y\,\mathrm{d}x + x\,\mathrm{d}y) &= -(h^*y)\,\mathrm{d}(h^*x) + (h^*x)\,\mathrm{d}(h^*y) \\ &= -\sin t\,\mathrm{d}(\cos t) + \cos t\,\mathrm{d}(\sin t) \\ &= \sin^2 t\,\mathrm{d}t + \cos^2 t\,\mathrm{d}t = \mathrm{d}t. \end{aligned} \tag{9.10}$$

So $h^*\omega = \mathrm{d}t$. Now,

$$F^*\omega = (h \circ i)^*\omega = i^*h^*\omega = i^*(\mathrm{d}t) = \mathrm{d}t. \tag{9.11}$$

$F : [0, 2\pi] \to S^1$ is a parametrization of $S^1 = F([0, 2\pi])$, with $[0, 2\pi]$ being the compact domain of integration in $\mathbb{R}$. Therefore,

$$\int_{S^1} \omega = \int_{[0,2\pi]} F^*\omega = \int_0^{2\pi} \mathrm{d}t = 2\pi. \tag{9.12}$$

Note that S^1 is a 1-dimensional smooth manifold. If there were any non-closed 1-form on S^1, its exterior derivative would be nonzero, resulting in a nontrivial 2-form on S^1. But there is no nontrivial 2-form on S^1 as it is a 1-dimension manifold. Hence, $\Omega^1(S^1) = Z^1(S^1)$. Now, consider the following linear map

$$\begin{aligned} \varphi : \Omega^1\left(S^1\right) = Z^1\left(S^1\right) &\to \mathbb{R}, \\ \alpha &\mapsto \int_{S^1} \alpha. \end{aligned}$$

By (9.12), $\varphi(\omega) = 2\pi \neq 0$. Choose any nonzero $r \in \mathbb{R}$ and take the one form $\frac{r}{2\pi}\omega \in \Omega^1(S^1)$. With these choices made, one immediately finds that

$$\varphi\left(\frac{r}{2\pi}\omega\right) = r, \tag{9.13}$$

by linearity of φ. In other words, for every $r \in \mathbb{R}$, there exists an element in $\Omega^1(S^1)$, namely $\frac{r}{2\pi}\omega$, such that $\varphi\left(\frac{r}{2\pi}\omega\right) = r$. Hence, $\varphi : Z^1(S^1) \to \mathbb{R}$ is surjective.

Now, suppose $\beta \in B^1(S^1)$. Hence, there exists $f \in C^\infty(S^1)$ such that $\beta = \mathrm{d}f$. Then

$$\int_{S^1} \beta = \int_{S^1} \mathrm{d}f = \int_{\partial S^1} f = \int_{\varnothing} f = 0. \tag{9.14}$$

So $\varphi(\beta) = 0$, and as a result, $\beta \in \operatorname{Ker}\varphi$. Hence, $B^1(S^1) \subseteq \operatorname{Ker}\varphi$. Let us now prove that $\operatorname{Ker}\varphi \subseteq B^1(S^1)$.

Since $\omega = -y\,\mathrm{d}x + x\,\mathrm{d}y$ is a nowhere vanishing smooth 1-form on S^1, any 1-form $\alpha \in \Omega^1(S^1)$ can be written as $\alpha = f\omega$, with $f \in C^\infty(S^1)$. Now suppose $\alpha = f\omega \in \operatorname{Ker}\varphi$. Also, let $\overline{f} = h^* f = f \circ h \in C^\infty(\mathbb{R})$. From (9.9) one easily finds that $h(t + 2\pi) = h(t)$, i.e. h is periodic with period 2π. Hence,

$$\overline{f}(t + 2\pi) = f(h(t + 2\pi)) = f(h(t)) = \overline{f}(t). \tag{9.15}$$

Therefore, $\overline{f}$ is also periodic of period 2π. Then

$$\begin{aligned} 0 = \varphi(\alpha) &= \int_{S^1} \alpha \\ &= \int_{[0,2\pi]} F^*\alpha = \int_{[0,2\pi]} F^*(f\omega) \\ &= \int_{[0,2\pi} F^*(f)\, F^*(\omega) \\ &= \int_0^{2\pi} (f \circ h \circ i)\ \mathrm{d}t \\ &= \int_0^{2\pi} \left(\overline{f} \circ i\right)\ \mathrm{d}t \\ &= \int_0^{2\pi} \overline{f}(t)\ \mathrm{d}t. \end{aligned} \tag{9.16}$$

Lemma 9.3

Suppose $\overline{f} \in C^\infty(\mathbb{R})$ is a periodic function of period 2π and $\int_0^{2\pi} \overline{f}(u)\ \mathrm{d}u = 0$. Then $\overline{f}\,\mathrm{d}t = \mathrm{d}g$ for a C^∞ periodic function $\overline{g}$ of period 2π on $\mathbb{R}$.

Proof. Define $\overline{g} \in \Omega^0(\mathbb{R})$ by

$$\overline{g}(t) = \int_0^t \overline{f}(u)\,\mathrm{d}u. \tag{9.17}$$

By the fundamental theorem of calculus, $\overline{g}' = \overline{f}$. Since by hypothesis $\int_0^{2\pi} \overline{f}(u)\ \mathrm{d}u = 0$, and $\overline{f}$ is 2π periodic, one has

$$\overline{g}(t + 2\pi) = \int_0^{t+2\pi} \overline{f}(u)\,\mathrm{d}u = \int_0^{2\pi} \overline{f}(u)\,\mathrm{d}u + \int_{2\pi}^{t+2\pi} \overline{f}(u)\,\mathrm{d}u \tag{9.18}$$

$$= \int_{2\pi}^{t+2\pi} \overline{f}(u)\,\mathrm{d}u = \int_0^t \overline{f}(u)\,\mathrm{d}u = \overline{g}(t), \tag{9.19}$$

proving that $\overline{g}$ is indeed periodic of period 2π on $\mathbb{R}$. Moreover,

$$\mathrm{d}\overline{g} = \overline{g}'\,\mathrm{d}t = \overline{f}\,\mathrm{d}t. \tag{9.20}$$

■

Proposition 9.4

For $k = 0, 1$, under the pullback map $h^* : \Omega^*\left(S^1\right) \to \Omega^*(\mathbb{R})$, smooth k-forms on S^1 are identified with smooth periodic k-forms of period 2π on $\mathbb{R}$.

Proof. If $f \in \Omega^0\left(S^1\right)$, then since $h : \mathbb{R} \to S^1$ is periodic of period 2π, the pullback $h^* f = f \circ h \in \Omega^0(\mathbb{R})$ is periodic of period 2π.

Conversely, suppose $\overline{f} \in \Omega^0(\mathbb{R})$ is periodic of period 2π. Let $p = h(t_0) \in S^1$. Then let (V, ψ) be a chart around p, with V being a small open circular arc, and ψ takes $(\cos x, \sin x)$ to x. Then with respect to the canonical basis on $T_{t_0}\mathbb{R}$ and $T_p S^1$, the matrix representation of $h_{*,t_0} : T_{t_0} \to T_p S^1$ is given by (using *Proposition 6.2.5* of DG1)

$$\left[\frac{\partial\left(x \circ h\right)}{\partial x}\left(t_0\right)\right] = [1]. \tag{9.21}$$

Therefore, $h_{*,t_0} : T_{t_0} \to T_p S^1$ is an isomorphism of vector spaces. As a result, by the *Inverse Function Theorem*, h is a local diffeomorphism. In other words, for every $p \in S^1$, there is a neighborhood U of p where $s : U \to (a, b) \subseteq \mathbb{R}$ is the smooth inverse of $h\big|_{(a,b)}$.

Then we define $f = \overline{f} \circ s$ on U. To show that f is well-defined, let s_1 and s_2 be two inverses of h over U. By the periodic properties of sine and cosine, $s_1 = s_2 + 2\pi n$ for some $n \in \mathbb{Z}$. Because $\overline{f}$ is periodic of period 2π, we have $\overline{f} \circ s_1 = \overline{f} \circ s_2$. This proves that f is well-defined on U. Moreover,

$$\overline{f} = f \circ s^{-1} = f \circ h = h^* f \quad \text{on } h^{-1}(U). \tag{9.22}$$

As p varies over S^1, we obtain a well-defined C^∞ function f on S^1 such that $\overline{f} = h^* f$. Thus, the image of $h^* : \Omega^0\left(S^1\right) \to \Omega^0(\mathbb{R})$ consists precisely of the C^∞ periodic functions of period 2π on $\mathbb{R}$.

As for 1-forms, note that $\Omega^1\left(S^1\right) = \Omega^0\left(S^1\right)\omega$, where $\omega = -y\,\mathrm{d}x + x\,\mathrm{d}y$, and $\Omega^1(\mathbb{R}) = \Omega^0(\mathbb{R})\mathrm{d}t$. The pullback $h^* : \Omega^1\left(S^1\right) \to \Omega^1(\mathbb{R})$ is given by $h^*(f\omega) = (h^* f)\,\mathrm{d}t$, so the image of $h^* : \Omega^1\left(S^1\right) \to \Omega^1(\mathbb{R})$ consists of C^∞ periodic 1-forms of period 2π. ■

Now let $\overline{g}$ the periodic function of period 2π on $\mathbb{R}$ as in Lemma 9.3. One then has $\overline{g} \in \operatorname{im} h^*$ for $k = 0$. Hence, there is a C^∞ function g on S^1 such that $h^* g = \overline{g}$. Then

$$\mathrm{d}\overline{g} = \mathrm{d}\left(h^* g\right) = h^*\left(\mathrm{d}g\right). \tag{9.23}$$

On the other hand,

$$\overline{f}\,\mathrm{d}t = h^*\left(f\right)\, h^*\left(\omega\right) = h^*\left(f\omega\right) = h^*\alpha. \tag{9.24}$$

Then from (9.20), (9.23), (9.24), one has

$$h^*\left(\mathrm{d}g\right) = h^*\alpha. \tag{9.25}$$

Now we claim that $h^* : \Omega^1\left(S^1\right) \to \Omega^1\left(\mathbb{R}\right)$ is injective. Let $\tau = j\omega \in \operatorname{Ker} h^*$, where $j \in C^\infty\left(S^1\right)$. Then $h^*\tau = h^* j\,\mathrm{d}t$. Since $\tau \in \operatorname{Ker} h^*$, we must have $h^* j = 0$. In other words, for any $t \in \mathbb{R}$,

$$0 = \left(h^* j\right)(t) = j\left(h\left(t\right)\right). \tag{9.26}$$

Since $h : \mathbb{R} \to S^1$ is surjective, $j = 0$ on S^1. Hence, $\tau = j\omega = 0$, proving the injectivity of $h^* : \Omega^1\left(S^1\right) \to \Omega^1\left(\mathbb{R}\right)$. Now, using (9.25) and the injectivity of h^*, we get

$$\alpha = \mathrm{d}g. \tag{9.27}$$

As a result, we get $\alpha \in B^1\left(S^1\right)$. Therefore, $\operatorname{Ker}\varphi \subseteq B^1\left(S^1\right)$. Earlier we proved $B^1\left(S^1\right) \subseteq \operatorname{Ker}\varphi$. So we have

$$B^1\left(S^1\right) = \operatorname{Ker}\varphi. \tag{9.28}$$

Now, $\varphi : Z^1\left(S^1\right) \to \mathbb{R}$ is a surjective linear map with kernel $B^1\left(S^1\right)$. Therefore, by the first isomorphism theorem,

$$\frac{Z^1\left(S^1\right)}{B^1\left(S^1\right)} = \frac{Z^1\left(S^1\right)}{\operatorname{Ker}\varphi} \cong \mathbb{R}. \tag{9.29}$$

So we have

$$H^k(S^1) \cong \begin{cases} \mathbb{R} & \text{for } k = 0, 1 \\ 0 & \text{for } k \geq 2. \end{cases} \tag{9.30}$$

§9.2 Diffeomorphism Invariance

Lemma 9.5

The pullback map $F^*: \Omega^*(M) \to \Omega^*(N)$ of the smooth map $F: N \to M$ sends closed forms to closed forms, and sends exact forms to exact forms.

Proof. Suppose $\omega \in \Omega^*(M)$ is closed. By the commutativity of F^* with d,

$$\mathrm{d}F^*\omega = F^*(\mathrm{d}\omega) = 0. \tag{9.31}$$

Hence, $F^*\omega$ is also closed. Next suppose $\omega = \mathrm{d}\tau$ is exact. Then

$$F^*\omega = F^*(\mathrm{d}\tau) = \mathrm{d}F^*\tau \tag{9.32}$$

Hence, $F^*\omega$ is exact. ■

If we restrict ourselves to k-forms only, then F^* sends $Z^k(M)$ to $Z^k(N)$, and $B^k(M)$ to $B^k(N)$, and thus induces a linear map of quotient spaces, denoted by $\left(F^{\#}\right)^k$:

$$\left(F^{\#}\right)^k : \frac{Z^k(M)}{B^k(M)} \to \frac{Z^k(N)}{B^k(N)}, \quad \left(F^{\#}\right)^k([\omega]) = [F^*\omega].$$

This map $\left(F^{\#}\right)^k : H^k(M) \to H^k(N)$ is a map in cohomology, called the pullback map in cohomology.

From Remark 5.2, we know that the pullback map $F^* : \Omega^*(M) \to \Omega^*(N)$ corresponding to the smooth map $F : N \to M$ is associated with a contravariant functor from the category **Man** of manifolds and smooth maps to the category **GrAlg** of graded algebras and graded algebra homomorphisms. The functorial properties (Theorem 5.7) of F^* on differential forms induce the same functorial properties for the induced map $\left(F^{\#}\right)^k$ in cohomology.

Theorem 9.6

The following holds:

(a) If $\mathbb{1}_M : M \to M$ is the identity map, then $\left((\mathbb{1}_M)^{\#}\right)^k : H^k(M) \to H^k(M)$ is also the identity map on the vector space $H^k(M)$.

(b) If $F : N \to M$ and $G : M \to P$ are smooth snaps of manifolds, then

$$\left((G \circ F)^{\#}\right)^k = \left(F^{\#}\right)^k \circ \left(G^{\#}\right)^k$$

Proof. We are going to use Theorem 5.7. Given a smooth map $F : N \to M$, the linear map $\left(F^{\#}\right)^k$: $H^k(M) \to H^k(N)$ is defined as

$$\left(F^{\#}\right)^k [\omega] = [F^*\omega]. \tag{9.33}$$

(a) Let $[\omega] \in H^k(M)$. Then

$$\left((\mathbb{1}_M)^{\#}\right)^k [\omega] = [(\mathbb{1}_M)^* \omega] = [\omega]. \tag{9.34}$$

Therefore,

$$\left((\mathbb{1}_M)^{\#}\right)^k = \mathbb{1}_{H^k(M)}. \tag{9.35}$$

(b) Suppose $[\omega] \in H^k(P)$. Then

$$\left((G \circ F)^{\#}\right)^k [\omega] = [(G \circ F)^* \omega] = [(F^* \circ G^*)\omega] = [F^*G^*\omega]. \tag{9.36}$$

On the other hand,

$$\left(F^{\#}\right)^{k} \circ \left(G^{\#}\right)^{k}[\omega] = \left(F^{\#}\right)^{k}[G^{*}\omega] = [F^{*}G^{*}\omega]\,. \tag{9.37}$$

Therefore,

$$\left((G \circ F)^{\#}\right)^{k} = \left(F^{\#}\right)^{k} \circ \left(G^{\#}\right)^{k}. \tag{9.38}$$

■

From Theorem 9.6, it immediately follows that $\left(H^{k}(-), \left(F^{\#}\right)^{k}\right)$ is a contravariant functor from the category **Man** of smooth manifolds and smooth maps to the category $\mathbf{Vect}_{\mathbb{R}}$ of real vector spaces and linear maps.

Now, if F is an isomorphism in the former category, i.e. a diffeomorphism in the category **Man**, then $\left(F^{\#}\right)^{k}$ will be an isomorphism in the latter category. In other words, $\left(F^{\#}\right)^{k} : H^{k}(M) \to H^{k}(N)$ will be an isomorphism of vector spaces.

§9.3 The Ring Structure on de Rham Cohomology

The wedge product $\wedge$ of differential forms endows the vector space $\Omega^{*}(M)$ with a product structure in cohomology: if $[\omega] \in H^{k}(M)$ and $[\tau] \in H^{l}(M)$, then we definee

$$[\omega] \wedge [\tau] = [\omega \wedge \tau]\,. \tag{9.39}$$

For the product in (9.39) to be well-defined, we need to check the following:

(i) The wedge product $\omega \wedge \tau$ is a closed form.

(ii) The class $[\omega \wedge \tau]$ is independent of the choice of the representative for $[\omega]$ or $[\tau]$. In other words, we need to show that $(\omega + \mathrm{d}\alpha) \wedge (\tau + \mathrm{d}\beta)$ is cohomologous to $\omega \wedge \tau$. This would prove that

$$[\omega + \mathrm{d}\alpha] \wedge [\tau + \mathrm{d}\beta] = [(\omega + \mathrm{d}\alpha) \wedge (\tau + \mathrm{d}\beta)] = [\omega \wedge \tau] = [\omega] \wedge [\tau]\,.$$

We have

$$(\omega + \mathrm{d}\alpha) \wedge (\tau + \mathrm{d}\beta) = \omega \wedge \tau + \omega \wedge \mathrm{d}\beta + \mathrm{d}\alpha \wedge \tau + \mathrm{d}\alpha \wedge d\beta. \tag{9.40}$$

Using the antiderivation property of d, we have

$$\mathrm{d}(\omega \wedge \beta) = \mathrm{d}\omega \wedge \beta + (-1)^{k}\omega \wedge \mathrm{d}\beta = (-1)^{k}\omega \wedge \mathrm{d}\beta, \tag{9.41}$$

since ω is closed. Therefore, $\omega \wedge \mathrm{d}\beta$ is an exact form. In a similar manner,

$$\mathrm{d}(\alpha \wedge (\tau + \mathrm{d}\beta)) = \mathrm{d}\alpha \wedge (\tau + d\beta) + (-1)^{k-1}\alpha \wedge \mathrm{d}(\tau + \mathrm{d}\beta) = \mathrm{d}\alpha \wedge (\tau + \mathrm{d}\beta). \tag{9.42}$$

Hence, $\alpha \wedge (\tau + \mathrm{d}\beta)$ is also exact. Therefore, $(\omega + \mathrm{d}\alpha) \wedge (\tau + \mathrm{d}\beta)$ is cohomologous to $\omega \wedge \tau$. Hence, $\wedge$ is well-defined.

Now, if M is a manifold of dimension n, we set

$$H^{*}(M) = \bigoplus_{k=0}^{n} H^{k}(M)\,. \tag{9.43}$$

(9.43) means that an element of $H^{*}(M)$ is uniquely a finite formal sum of cohomology classes in $H^{k}(M)$ as k varies:

$$\alpha = \alpha_0 + \cdots + \alpha_n, \tag{9.44}$$

with $\alpha_k \in H^{k}(M)$. Now, one can easily verify that with the formal addition and wedge product, $H^{*}(M)$ satisfies all the properties of a ring. We call this ring the **cohomology ring** of M.

A ring $(A, +, \times)$ is **graded** if it can be written as a direct sum $A = \bigoplus_{k=0}^{\infty} A^k$ so that the ring multiplication $\times$ sends $A^k \times A^l$ to A^{k+l}. A graded ring $A = \bigoplus_{k=0}^{\infty} A^k$ is said to be **anticommutative** if for all $a \in A^k$ and $b \in A^l$,

$$a \times b = (-1)^{kl}\, b \times a. \tag{9.45}$$

Since the wedge product of differential forms is defined pointwise, i.e. for $\omega, \tau \in \Omega^*(M)$,

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p,$$

we have $\omega \wedge \tau = (-1)^{kl}\, \tau \wedge \omega$ whenever $\omega \in \Omega^k(M)$ and $\tau \in \Omega^l(M)$. This way, $H^*(M)$ also becomes an anticommutative graded ring. Indeed, for $[\omega] \in H^k(M)$ and $[\tau] \in H^l(M)$,

$$\begin{aligned} [\omega] \wedge [\tau] = [\omega \wedge \tau] &= \left[(-1)^{kl}\, \tau \wedge \omega\right] \\ &= (-1)^{kl}\, [\tau \wedge \omega] = (-1)^{kl}\, [\tau] \wedge [\omega]\,. \end{aligned} \tag{9.46}$$

This way $H^*(M)$ becomes an anticommutative graded ring. Since $H^*(M)$ is also a real vector space, it is, in fact, an anticommutative graded algebra over $\mathbb{R}$.

Suppose $F : N \to M$ is a smooth map of manifolds. Since $F^*(\omega \wedge \tau) = F^*\omega \wedge F^*\tau$ for $\omega, \tau \in \Omega^*(M)$, the linear map

$$F^{\#} : H^*(M) \to H^*(N),$$

is a ring homomorphism. Then $\left(H^*(-), F^{\#}\right)$ becomes a contravariant functor from the category of smooth manifolds and smooth maps to the category of anticommutative graded rings and ring homomorphisms. If $F : N \to M$ is an isomorphism in the former category, i.e. if $F : N \to M$ is a diffeomorphism, then it is an isomorphism in the latter category as well, i.e. $F^{\#} : H^*(M) \to H^*(N)$ is a ring isomorphism.

10 The Long Exact Sequence of Cohomology

Definition 10.1 (Cochain complex). A **cochain complex** $\mathcal{C}$ is a collection of vector spaces $\left\{C^k\right\}_{k\in\mathbb{Z}}$ together with a sequence of linear maps $d_k : C^k \to C^k$

$$\cdots \longrightarrow C^{-1} \xrightarrow{d_{-1}} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \longrightarrow \cdots$$

such that for all $k \in \mathbb{Z}$,

$$d_k \circ d_{k-1} = 0. \tag{10.1}$$

We call the linear maps in the collection $\{d_k\}_{k\in\mathbb{Z}}$, the **differentials** of the cochain complex $\mathcal{C}$.

The vector space $\Omega^*(M)$ of differential forms on a manifold M together with the exterior derivative d is a cochain complex, called the de Rham complex of M :

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M) \xrightarrow{\mathrm{d}} \Omega^3(M) \longrightarrow \cdots,$$

and $\mathrm{d} \circ \mathrm{d} = 0$. Many of the results on the de Rham cohomology of a manifold depend not on the topological properties of the manifold but on the algebraic properties of the de Rham complex. To better study de Ream cohomology, it is useful to isolate these algebraic properties that we do in this chapter.

§10.1 Exact Sequences

Definition 10.2 (Exact sequence). A sequence of homomorphism of vector spaces $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be **exact** at B if $\operatorname{im} f = \operatorname{Ker} g$. A sequence of homomorphisms

$$A^0 \xrightarrow{f_0} A^1 \xrightarrow{f_1} A^2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A^n$$

that is exact at every term except the first and the last term is called an **exact sequence**. A fiver term exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is said to be **short exact**.

Remark 10.1. When $A = 0$ in the three-term exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$, i.e. $0 \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $\operatorname{Ker} g = \operatorname{im} f = 0$, so that g is injective.

Similarly, when $C = 0$ in the three-term exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$, i.e. $A \xrightarrow{f} B \xrightarrow{g} 0$ is exact if and only if $\operatorname{im} f = \operatorname{Ker} g = B$, so that f is surjective.

Proposition 10.1

Suppose $A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence. Then

(i) the map f is surjective if and only if g is the zero map;

(ii) the map g is injective if and only if f is the zero map.

Proof. (i) Since the sequence is exact, we have $\operatorname{im} f = \operatorname{Ker} g$. $f : A \to B$ is surjective if and only if $\operatorname{im} f = B$. Therefore, the surjectivity of f is equivalent to $\operatorname{Ker} g = B$. $\operatorname{Ker} g = B$ means it takes all of B to $0 \in C$, i.e. g is the zero map.

(ii) $g : B \to C$ is injective if and only if $\operatorname{Ker} g = 0$. Therefore, the injectivity of g is equivalent to $\operatorname{im} f = 0$. $\operatorname{im} f = 0$ means it takes all of A to $0 \in B$, i.e. f is the zero map.

■

Proposition 10.2

The following hold:

(i) The 4-term sequence

$$0 \longrightarrow A \xrightarrow{\;f\;} B \longrightarrow 0$$

is exact if and only if $f : A \to B$ is an isomorphism.

(ii) If the following

$$A \xrightarrow{\;f\;} B \longrightarrow C \longrightarrow 0$$

is an exact sequence of vector spaces, then there is a linear isomorphism

$$C \cong \operatorname{Coker} f := \frac{B}{\operatorname{im} f}.$$

Proof. (i) Suppose that the 4-term sequence

$$0 \longrightarrow A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} 0$$

is exact. Then by Proposition 10.1, by the exactness of $0 \to A \xrightarrow{f} B$, we get f is injective. Again, using Proposition 10.1 and the exactness of $A \xrightarrow{f} B \to 0$, f is surjective. Therefore, f is bijective. The inverse of a bijective linear map is also linear. Hence, f is an isomorphism.

Conversely, suppose $f : A \to B$ is an isomorphism of vector spaces. Hence, f is injective and surjective. Since f is injective, we have

$$\operatorname{Ker} f = 0 = \operatorname{im}(0 \to A)\,. \tag{10.2}$$

Again, f is surjective, so

$$\operatorname{im} f = B = \operatorname{Ker}(B \to 0)\,. \tag{10.3}$$

Therefore, the 4-term sequence

$$0 \longrightarrow A \xrightarrow{\;f\;} B \longrightarrow 0$$

is exact.

(ii) Suppose the 4-term sequence

$$0 \longrightarrow A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} 0$$

is an exact sequence of vector spaces. By Proposition 10.1, g is surjective. By exactness of this sequence at B, $\operatorname{im} f = \operatorname{Ker} g$. Now, applying the first isomorphism theorem for the surjective linear map $g : B \to C$, we get an an isomorphism

$$C = \operatorname{im} g \cong \frac{B}{\operatorname{Ker} g} = \frac{B}{\operatorname{im} f} = \operatorname{Coker} f. \tag{10.4}$$

■

§10.2 Cohomology of cochain complexes

Recall that a cochain complex $\mathcal{C}$ is a collection of vector spaces $\left\{C^k\right\}_{k\in\mathbb{Z}}$ together with a sequence of linear maps $d_k : C^k \to C^k$

$$\cdots \longrightarrow C^{k-1} \xrightarrow{d_{k-1}} C^k \xrightarrow{d_k} C^{k+1} \longrightarrow \cdots$$

such that for all $k \in \mathbb{Z}$, $d_k \circ d_{k-1} = 0$. This implies that

$$\operatorname{im} d_{k-1} \subseteq \operatorname{Ker} d_k. \tag{10.5}$$

One can, therefore, form the quotient vector space

$$H^k\left(\mathcal{C}\right) := \frac{\operatorname{Ker} d_k}{\operatorname{im} d_{k-1}}, \tag{10.6}$$

which is called the k-th **cohomology vector space** of the cochain complex $\mathcal{C}$. It is a measure of the extent to which the cochain complex $\mathcal{C}$ fails to he exact at C^k. Elements of the vector space C^k are called cochains of degree k or k-cochains. A k-cochain in $\operatorname{Ker} d_k$ is called a k-cocycle; a k-cochain in $\operatorname{im} d_{k-1}$ is called a k-coboundary. The equivalence, class $[c] \in\in H^k(\mathcal{C})$ of a k-cocycle $c \in \operatorname{Ker} d_k$ is called its cohomology class. We denote these 2 subspaces of C^k by $Z^k(\mathcal{C})$ (subspace of k-cocycles) and by $B^k(\mathcal{C})$ (subspace of k-coboundaries).

Example 10.1. In the de Rham complex of a manifold M, a cocylce is a closed form and a coboundary is an exact form.

Definition 10.3 (Cochain map)**.** If $\mathcal{A}$ and $\mathcal{B}$ are 2 cochain complexes with differentials $\left\{d_k^{\mathcal{A}}\right\}_{k\in\mathbb{Z}}$ and $\left\{d_k^{\mathcal{B}}\right\}_{k\in\mathbb{Z}}$, respectively. A **cochain map** $\varphi : \mathcal{A} \to \mathcal{B}$ is a collection of linear maps $\varphi_k : A^k \to B^k$ such that

$$d_k^{\mathcal{B}} \circ \varphi_k = \varphi_{k+1} \circ d_k^{\mathcal{A}}. \tag{10.7}$$

In other words, the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A^{k-1} & \xrightarrow{d_{k-1}^{\mathcal{A}}} & A^k & \xrightarrow{d_k^{\mathcal{A}}} & A^{k+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \varphi_{k-1}} & & \downarrow{\scriptstyle \varphi_k} & & \downarrow{\scriptstyle \varphi_{k+1}} & & \\
\cdots & \longrightarrow & B^{k-1} & \xrightarrow[d_{k-1}^{\mathcal{B}}]{} & B^k & \xrightarrow[d_k^{\mathcal{B}}]{} & B^{k+1} & \longrightarrow & \cdots
\end{array}$$

Observe that a cochain map $\varphi : \mathcal{A} \to \mathcal{B}$ takes cocycles to cocycles and coboundaries to coboundaries. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A^{k-1} & \xrightarrow{d_{k-1}^{\mathcal{A}}} & A^k & \xrightarrow{d_k^{\mathcal{A}}} & A^{k+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \varphi_{k-1}} & & \downarrow{\scriptstyle \varphi_k} & & \downarrow{\scriptstyle \varphi_{k+1}} & & \\
\cdots & \longrightarrow & B^{k-1} & \xrightarrow[d_{k-1}^{\mathcal{B}}]{} & B^k & \xrightarrow[d_k^{\mathcal{B}}]{} & B^{k+1} & \longrightarrow & \cdots
\end{array} \tag{10.8}$$

(i) For $a \in Z^k\left(\mathcal{A}\right)$, $d_k^{\mathcal{A}} a = 0$. Then by the commutativity of the right hand square in (10.8),

$$d_k^{\mathcal{B}}\left(\varphi_k\left(a\right)\right) = \varphi_{k+1}\left(d_k^{\mathcal{A}} a\right) = 0. \tag{10.9}$$

Therefore, $\varphi_k\left(a\right) \in \operatorname{Ker} d_k^{\mathcal{B}} = Z^k\left(\mathcal{B}\right)$. In other words, the cochain map $\varphi : \mathcal{A} \to \mathcal{B}$ takes cocycles to cocycles.

(ii) Suppose $a \in B^k(\mathcal{A})$. Then $a = d^{\mathcal{A}}_{k-1} a'$ for some $a' \in A^{k-1}$. Then by the commutativity of the left hand square in (10.8),

$$\varphi_k(a) = \varphi_k\left(d^{\mathcal{A}}_{k-1} a'\right) = d^{\mathcal{B}}_{k-1}\left(\varphi_{k-1} a'\right). \tag{10.10}$$

Therefore, $\varphi_k(a) \in \operatorname{im} d^{\mathcal{B}}_{k-1} = B^k(\mathcal{B})$. In other words, the cochain map $\varphi : \mathcal{A} \to \mathcal{B}$ takes coboundaries to coboundaries.

Hence, we see that the coochain map $\varphi : \mathcal{A} \to \mathcal{B}$ naturally induces a linear map in cohomology:

$$\begin{aligned} (\varphi^*)^k : H^k(\mathcal{A}) &\to H^k(\mathcal{B}), \\ [a] &\mapsto [\varphi_k(a)]. \end{aligned} \tag{10.11}$$

Example 10.2. For a smooth map $F : N \to M$ between manifolds, the pullback map $F^* : \Omega^*(M) \to \Omega^*(N)$ on differential forms is a cochain map, because F^* commutes with d. By the discussion above, there is an induced map $\left(F^{\#}\right)^k : H^k(M) \to H^k(N)$ in cohomology.

§10.3 Zig-Zag Lemma

A sequence of cochain complexes

$$0 \longrightarrow \mathcal{A} \xrightarrow{\ i\ } \mathcal{B} \xrightarrow{\ j\ } \mathcal{C} \longrightarrow 0$$

is **short exact** if i and j are cochain maps, and for each k,

$$0 \longrightarrow A^k \xrightarrow{\ i_k\ } B^k \xrightarrow{\ j_k\ } C^k \longrightarrow 0$$

is a short exact sequence of vector spaces. In other words, the following is a commutative diagram with exact rows, for each k:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^{k+1} & \xrightarrow{i_{k+1}} & B^{k+1} & \xrightarrow{j_{k+1}} & C^{k+1} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle d^{\mathcal{A}}_k} & & \uparrow{\scriptstyle d^{\mathcal{B}}_k} & & \uparrow{\scriptstyle d^{\mathcal{C}}_k} & & \\ 0 & \longrightarrow & A^k & \xrightarrow[i_k]{} & B^k & \xrightarrow[j_k]{} & C^k & \longrightarrow & 0 \end{array} \tag{10.12}$$

Given a short exact sequence as above, we can constricy a linear map

$$(d^*)_k : H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A}),$$

called the **connecting homomorphism** as follows: consider the short exact sequences in dimensions k and $k+1$ associated with the short exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ of cochain complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^{k+1} & \xrightarrow{i_{k+1}} & B^{k+1} & \xrightarrow{j_{k+1}} & C^{k+1} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle d^{\mathcal{A}}_k} & & \uparrow{\scriptstyle d^{\mathcal{B}}_k} & & \uparrow{\scriptstyle d^{\mathcal{C}}_k} & & \\ 0 & \longrightarrow & A^k & \xrightarrow[i_k]{} & B^k & \xrightarrow[j_k]{} & C^k & \longrightarrow & 0 \end{array} \tag{10.13}$$

We start with $[c] \in H^k(\mathcal{C})$, for some $c \in \operatorname{Ker} d^{\mathcal{C}}_k \subseteq C^k$. By the exactness of the bottom row, we have that j_k is surjective. So there is some $b \in B^k$ such that $j_k(b) = c$. By the commutativity of the right square,

$$j_{k+1}\left(d^{\mathcal{B}}_k b\right) = d^{\mathcal{C}}_k(j_k b) = d^{\mathcal{C}}_k(c) = 0. \tag{10.14}$$

Therefore, $d^{\mathcal{B}}_k b \in \operatorname{Ker} j_{k+1} = \operatorname{im} i_{k+1}$. So $d^{\mathcal{B}}_k b = i_{k+1}(a)$ for some $a \in A^{k+1}$. Now consider the diagram (10.13) for $k+1$ in place of k.

$$\begin{CD} 0 @>>> A^{k+2} @>i_{k+2}>> B^{k+2} @>j_{k+2}>> C^{k+2} @>>> 0 \\ @. @A{d^{\mathcal{A}}_{k+1}}AA @A{d^{\mathcal{B}}_{k+1}}AA @A{d^{\mathcal{C}}_{k+1}}AA @. \\ 0 @>>> A^{k+1} @>>i_{k+1}> B^{k+1} @>>j_{k+1}> C^{k+1} @>>> 0 \end{CD} \tag{10.15}$$

Now,

$$i_{k+2}\left(d^{\mathcal{A}}_{k+1} a\right) = d^{\mathcal{B}}_{k+1}\left(i_{k+1} a\right) = d^{\mathcal{B}}_{k+1}\left(d^{\mathcal{B}}_k b\right) = 0. \tag{10.16}$$

So $d^{\mathcal{A}}_{k+1} a \in \operatorname{Ker} i_{k+2}$. But i_{k+2} is injective by the exactness of the top row of (10.15). Therefore,

$$d^{\mathcal{A}}_{k+1} a = 0. \tag{10.17}$$

So we define

$$(d^*)_k [c] = [a]. \tag{10.18}$$

The recipe for defining the connecting homomorphism $(d^*)_k$ is best summarized by the following Zig-Zag diagram:

$$\begin{array}{ccccc} a & \xrightarrow{i_{k+1}} & d^{\mathcal{B}}_k b & & \\ & & \Big\uparrow {\scriptstyle d^{\mathcal{B}}_k} & & \\ & & b & \xrightarrow{j_k} & c \end{array}$$

Note that there were choices involved in this definition. We chose the cocycle c to represent the cohomology class $[c]$. One could've chosen a cohomologous cocycle c' representing the same cohomology class $[c]$. Furthermore, we chose an element $b \in B^k$ such that $j_k(b) = c$ holds. Since j_k is surjective, and not necessarily injective, the choice for b is not unique. So we need to show that this definition of $(d^*)_k : H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$ is well-defined.

Let $[c] = [c']$. Then

$$c - c' = d_{k-1} c'', \tag{10.19}$$

for some $c'' \in C^{k-1}$. As before, we choose some $b' \in B^k$ such that $j_k(b') = c'$, and then finally $d^{\mathcal{B}}_k b' = i_{k+1}(a')$. We need to show that $[a] = [a']$.

$$\begin{CD} 0 @>>> A^{k+1} @>i_{k+1}>> B^{k+1} @>j_{k+1}>> C^{k+1} @>>> 0 \\ @. @A{d^{\mathcal{A}}_{k}}AA @A{d^{\mathcal{B}}_{k}}AA @A{d^{\mathcal{C}}_{k}}AA @. \\ 0 @>>> A^{k} @>>i_{k}> B^{k} @>>j_{k}> C^{k} @>>> 0 \\ @. @A{d^{\mathcal{A}}_{k-1}}AA @A{d^{\mathcal{B}}_{k-1}}AA @A{d^{\mathcal{C}}_{k-1}}AA @. \\ 0 @>>> A^{k-1} @>>i_{k-1}> B^{k-1} @>>j_{k-1}> C^{k-1} @>>> 0 \end{CD}$$

Since j_{k-1} is surjective, there exists $b'' \in B^{k-1}$ such that $j_{k-1}(b'') = c''$. Then

$$\begin{aligned} j_k\left(b - b' - d^{\mathcal{B}}_{k-1} b''\right) &= j_k(b) - j_k(b') - j_k\left(d^{\mathcal{B}}_{k-1} b''\right) \\ &= j_k(b) - j_k(b') - d^{\mathcal{C}}_{k-1}\left(j_{k-1} b''\right) \\ &= c - c' - d_{k-1} c'' = 0. \end{aligned} \tag{10.20}$$

As a result, $b - b' - d_{k-1}^{\mathcal{B}} b'' \in \operatorname{Ker} j_k = \operatorname{im} i_k$. So

$$b - b' - d_{k-1}^{\mathcal{B}} b'' = i_k \left(a''\right) \tag{10.21}$$

for some $a'' \in A^k$. Now,

$$\begin{aligned} i_{k+1}\left(d_k^{\mathcal{A}} a''\right) &= d_k^{\mathcal{B}}\left(i_k a''\right) = d_k^{\mathcal{B}}\left(b - b' - d_{k-1}^{\mathcal{B}} b''\right) \\ &= d_k^{\mathcal{B}}(b) - d_k^{\mathcal{B}}(b') = i_{k+1}(a) - i_{k+1}(a') \\ &= i_{k+1}\left(a - a'\right). \end{aligned} \tag{10.22}$$

Since i_{k+1} is injective, we have

$$a - a' = d_k^{\mathcal{A}} a''. \tag{10.23}$$

In other words, $[a] = [a']$, i.e. $(d^*)_k$ is well-defined.

It's easy to show that $(d^*)_k$ is linear. Given $[c], [c'] \in H^k(\mathcal{C})$ and $\alpha \in \mathbb{R}$, $[c] + \alpha [c'] = [c + \alpha c']$. Suppose $c = j_k(b)$, $c' = j_k(b')$; and $d_k^{\mathcal{B}} b = i_{k+1}(a)$, $d_k^{\mathcal{B}} b' = i_{k+1}(a')$. Then

$$(d^*)_k [c] = [a], \text{ and } (d^*)_k [c'] = [a']. \tag{10.24}$$

Now, by the linearity of j_k,

$$j_k\left(b + \alpha b'\right) = j_k(b) + \alpha j_k\left(b'\right) = c + \alpha c'. \tag{10.25}$$

Furthermore, using the linearity of $d_k^{\mathcal{B}}$ and i_{k+1},

$$i_{k+1}\left(a + \alpha a'\right) = i_{k+1}(a) + \alpha i_{k+1}\left(a'\right) = d_k^{\mathcal{B}} b + \alpha d_k^{\mathcal{B}} b' = d_k^{\mathcal{B}}\left(b + \alpha b'\right). \tag{10.26}$$

Therefore,

$$(d^*)_k \left[c + \alpha c'\right] = \left[a + \alpha a'\right] = [a] + \alpha \left[a'\right]. \tag{10.27}$$

In other words,

$$(d^*)_k \left([c] + \alpha \left[c'\right]\right) = (d^*)_k [c] + \alpha (d^*)_k [c]. \tag{10.28}$$

Hence, $(d^*)_k$ is a linear map.

Theorem 10.3 (The Zig-Zag Lemma)

Given a short exact sequence of cochain complexes

$$0 \longrightarrow \mathcal{A} \xrightarrow{\ i\ } \mathcal{B} \xrightarrow{\ j\ } \mathcal{C} \longrightarrow 0,$$

one has a long exact sequence in cohomology:

$$\begin{array}{ccccc} H^{k+1}(\mathcal{A}) & \xrightarrow{(i^*)^{k+1}} & \cdots & & \\ & \nwarrow^{(d^*)_k} & & & \\ H^k(\mathcal{A}) & \xrightarrow{(i^*)^k} & H^k(\mathcal{B}) & \xrightarrow{(j^*)^k} & H^k(\mathcal{C}) \\ & \nwarrow^{(d^*)_{k-1}} & & & \\ & & \cdots & \xrightarrow{(j^*)^{k-1}} & H^{k-1}(\mathcal{C}), \end{array} \tag{10.29}$$

where $(i^*)^k$ ad $(j^*)^k$ are the maps in cohomology induced from the cochain maps i and j; and $(d^*)_k$ is the connecting homomorphism defined earlier.

Proof. To prove this theorem, one needs to check the exactness of the above sequence (10.29) at $H^k(\mathcal{A})$, $H^k(\mathcal{B})$ and $H^k(\mathcal{C})$ for each k. We shall make use of the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^{k+1} & \xrightarrow{i_{k+1}} & B^{k+1} & \xrightarrow{j_{k+1}} & C^{k+1} & \longrightarrow & 0 \\ & & \Big\uparrow d_k^{\mathcal{A}} & & \Big\uparrow d_k^{\mathcal{B}} & & \Big\uparrow d_k^{\mathcal{C}} & & \\ 0 & \longrightarrow & A^{k} & \xrightarrow{i_{k}} & B^{k} & \xrightarrow{j_{k}} & C^{k} & \longrightarrow & 0 \\ & & \Big\uparrow d_{k-1}^{\mathcal{A}} & & \Big\uparrow d_{k-1}^{\mathcal{B}} & & \Big\uparrow d_{k-1}^{\mathcal{C}} & & \\ 0 & \longrightarrow & A^{k-1} & \xrightarrow{i_{k-1}} & B^{k-1} & \xrightarrow{j_{k-1}} & C^{k-1} & \longrightarrow & 0 \end{array} \tag{10.30}$$

Exactness at $H^k(\mathcal{A})$: Let $[a] \in \mathrm{Ker}\,(i^*)_k$.

$$(i^*)_k\,[a] = [i_k\,(a)] = [0] \in H^k\,(\mathcal{B})\,. \tag{10.31}$$

In other words,

$$i_k\,(a) = d_{k-1}^{\mathcal{B}} b, \tag{10.32}$$

for some $b \in B^{k-1}$. Let $c = j_{k-1}\,(b)$. Then

$$d_{k-1}^{\mathcal{C}}\,(c) = d_{k-1}^{\mathcal{C}}\,(j_{k-1}\,(b)) = j_k\left(d_{k-1}^{\mathcal{B}} b\right) = j_k\,(i_k\,(a)) = 0, \tag{10.33}$$

since $j_k \circ i_k = 0$. Therefore, $c \in \mathrm{Ker}\,d_{k-1}^{\mathcal{C}}$, i.e. $[c] \in H^{k-1}\,(\mathcal{C})$. $b \in B^{k-1}$ is such that $j_{k-1}\,(b) = c$. Furthermore, $d_{k-1}^{\mathcal{B}} b = i_k\,(a)$.

$$\begin{array}{ccc} a & \overset{i_k}{\rightarrowtail} & d_{k-1}^{\mathcal{B}} b \\ & & \Big\uparrow d_{k-1}^{\mathcal{B}} \\ & & b \overset{j_{k-1}}{\twoheadrightarrow} c \end{array}$$

Therefore, $[a] = (d^*)_{k-1}\,[c]$. In other words,

$$\mathrm{Ker}\,(i^*)_k \subseteq \mathrm{im}\,(d^*)_{k-1}\,. \tag{10.34}$$

Now suppose $[a] \in \mathrm{im}\,(d^*)_{k-1}$, i.e. $[a] = (d^*)_{k-1}\,[c]$ for some $[c] \in H^{k-1}\,(\mathcal{C})$. Then $c = j_{k-1}\,(b)$ for some $b \in B^{k-1}$, and $i_k\,(a) = d_{k-1}^{\mathcal{B}} b$. Now,

$$(i^*)^k\,[a] = [i_k\,(a)] = \left[d_{k-1}^{\mathcal{B}} b\right] = [0] \in H^k\,(\mathcal{B}) = \frac{\mathrm{Ker}\,d_k^{\mathcal{B}}}{\mathrm{im}\,d_{k-1}^{\mathcal{B}}}. \tag{10.35}$$

Therefore, $[a] \in \mathrm{Ker}\,(i^*)^k$. So

$$\mathrm{im}\,(d^*)_{k-1} \subseteq \mathrm{Ker}\,(i^*)_k\,. \tag{10.36}$$

As a result of (10.34) and (10.36), we have

$$\mathrm{Ker}\,(i^*)_k = \mathrm{im}\,(d^*)_{k-1}\,. \tag{10.37}$$

So (10.29) is exact at $H^k\,(\mathcal{A})$.

Exactness at $H^k\,(\mathcal{B})$: Given $[a] \in H^k\,(\mathcal{A})$,

$$(j^*)^k \circ (i^*)^k\,[a] = (j^*)^k\,[i_k\,(a)] = [j_k\,(i_k\,(a))] = 0, \tag{10.38}$$

since $j_k \circ i_k = 0$. Therefore,

$$\mathrm{im}\,(i^*)^k \subseteq \mathrm{Ker}\,(j^*)^k\,. \tag{10.39}$$

Now, suppose $[b] \in \operatorname{Ker}(j^*)^k$. Then

$$(j^*)^k [b] = [j_k b] = [0] \in H^k(\mathcal{C}). \tag{10.40}$$

So we have

$$j_k(b) = d^{\mathcal{C}}_{k-1} c \tag{10.41}$$

for some $c \in C^{k-1}$. Since j_{k-1} is surjective, we have

$$c = j_{k-1}(b') \tag{10.42}$$

for some $b' \in B^{k-1}$. Then

$$\begin{aligned} j_k\left(b - d^{\mathcal{B}}_{k-1} b'\right) &= j_k(b) - j_k\left(d^{\mathcal{B}}_{k-1} b'\right) \\ &= d^{\mathcal{C}}_{k-1} c - d^{\mathcal{C}}_{k-1}(j_{k-1} b') \\ &= d^{\mathcal{C}}_{k-1} c - d^{\mathcal{C}}_{k-1} c = 0. \end{aligned} \tag{10.43}$$

So $b - d^{\mathcal{B}}_{k-1} b' \in \operatorname{Ker} j_k = \operatorname{im} i_k$, i.e. $b - d^{\mathcal{B}}_{k-1} b' = i_k(a)$ for some $a \in A^k$. Now,

$$i_{k+1}\left(d^{\mathcal{A}}_k a\right) = d^{\mathcal{B}}_k(i_k a) = d^{\mathcal{B}}_k\left(b - d^{\mathcal{B}}_{k-1} b'\right) = d^{\mathcal{B}}_k b = 0, \tag{10.44}$$

since $[b] \in H^k(\mathcal{B})$. Now, i_{k+1} is injective. As a result, $d^{\mathcal{A}}_k a = 0$. Now,

$$(i^*)^k [a] = [i_k(a)] = \left[b - d^{\mathcal{B}}_{k-1} b'\right] = [b], \tag{10.45}$$

i.e. $[b] \in \operatorname{im}(i^*)^k$. So

$$\operatorname{Ker}(j^*)^k \subseteq \operatorname{im}(i^*)^k. \tag{10.46}$$

As a result of (10.39) and (10.46), we have

$$\operatorname{Ker}(j^*)^k = \operatorname{im}(i^*)^k. \tag{10.47}$$

Hence, (10.29) is exact at $H^k(\mathcal{B})$.

Exactness at $H^k(\mathcal{C})$: First we prove that $\operatorname{im}(j^*)^k \subseteq \operatorname{Ker}(d^*)_k$. For $[b] \in H^k(\mathcal{B})$, we have

$$(d^*)_k\left((j^*)^k [b]\right) = (d^*)_k [j_k(b)]. \tag{10.48}$$

In the recipe for defining $(d^*)_k : H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$, we can choose the element $b \in B^k$ that maps to $j_k(b)$. Then $d^{\mathcal{B}}_k b \in B^{k+1}$. Since $[b]$ is a k-th cohomology class, $b \in \operatorname{Ker} d^{\mathcal{B}}_k$. Therefore, $d^{\mathcal{B}}_k b = 0$. Following the Zig-Zag diagram,

$$\begin{array}{ccc} 0 & \xrightarrow{\;i_{k+1}\;} & d^{\mathcal{B}}_k b = 0 & & \\ & & \uparrow{\scriptstyle d^{\mathcal{B}}_k} & & \\ & & b & \xrightarrow{\;j_k\;} & j_k(b), \end{array}$$

we see that since $i_{k+1}(0) = 0 = d^{\mathcal{B}}_k b$, by the definition of $(d^*)^k$, we must have

$$(d^*)^k [j_k(b)] = 0. \tag{10.49}$$

Therefore, $(j^*)^k [b] \in \operatorname{Ker}(d^*)_k$, proving the inclusion

$$\operatorname{im}(j^*)^k \subseteq \operatorname{Ker}(d^*)_k. \tag{10.50}$$

Now, let $[c] \in \operatorname{Ker}(d^*)_k \subseteq H^k(\mathcal{C})$. Then

$$(d^*)_k [c] = [a] = 0. \tag{10.51}$$

So a is a $(k+1)$-coboundary, i.e.

$$a = d^{\mathcal{A}}_k a', \tag{10.52}$$

for some $a' \in A^k$. The calculation for $(d^*)_k [c]$ can be representative by the following Zig-Zag diagram:

$$\begin{array}{ccccc} d_k^{\mathcal{B}} a' = a & \xrightarrow{i_{k+1}} & d_k^{\mathcal{B}} b & & \\ \Big\uparrow {\scriptstyle d_k^{\mathcal{A}}} & & \Big\uparrow {\scriptstyle d_k^{\mathcal{B}}} & & \\ a' & & b & \xrightarrow{j_k} & c = j_k(b) \end{array}$$

Here b is an element in B^k such that $j_k(b) = c$, and $i_{k+1}(a) = d_k^{\mathcal{B}} b$. $a' \in A^k$, $i_k : A^k \to B^k$. So both $i_k(a')$ and b are in B^k. Now,

$$\begin{aligned} d_k^{\mathcal{B}}\left(b - i_k\left(a'\right)\right) &= d_k^{\mathcal{B}} b - d_k^{\mathcal{B}}\left(i_k\left(a'\right)\right) \\ &= d_k^{\mathcal{B}} b - i_{k+1}\left(d_k^{\mathcal{A}} a'\right) \\ &= d_k^{\mathcal{B}} b - i_{k+1}(a) = 0. \end{aligned} \tag{10.53}$$

Therefore, $b - i_k(a') \in \operatorname{Ker} d_k^{\mathcal{B}}$, i.e. $b - i_k(a')$ is cocycle in B^k. Now,

$$j_k\left(b - i_k\left(a'\right)\right) = j_k(b) - j_k\left(i_k\left(a'\right)\right) = j_k(b) = c, \tag{10.54}$$

since $j_k \circ i_k = 0$ by the exactness of $0 \to A^k \xrightarrow{i_k} B^k \xrightarrow{j_k} C \to 0$. Therefore, $b - i_k(a')$ is a cocycle in B^k that gets mapped to c under j_k. Therefore,

$$[c] = \left[j_k\left(b - i_k\left(a'\right)\right)\right] = \left(j^*\right)^k\left[b - i_k\left(a'\right)\right]. \tag{10.55}$$

So $[c] \in \operatorname{im}(j^*)^k$. As a result,

$$\operatorname{Ker}\left(d^*\right)_k \subseteq \operatorname{im}\left(j^*\right)^k. \tag{10.56}$$

Combining (10.50) and (10.56), we have

$$\operatorname{im}\left(j^*\right)^k = \operatorname{Ker}\left(d^*\right)_k, \tag{10.57}$$

proving the exactness of (10.29) at $H^k(\mathcal{C})$. ■

Corollary 10.4 (The Snake Lemma)

A commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^1 & \longrightarrow & B^1 & \longrightarrow & C^1 & \longrightarrow & 0 \\ & & \Big\uparrow {\scriptstyle \alpha} & & \Big\uparrow {\scriptstyle \beta} & & \Big\uparrow {\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A^0 & \longrightarrow & B^0 & \longrightarrow & C^0 & \longrightarrow & 0 \end{array}$$

induces a long exact sequence

$$\begin{array}{ccccccccc} & & \operatorname{cok}\alpha & \longrightarrow & \operatorname{cok}\beta & \longrightarrow & \operatorname{cok}\gamma & \longrightarrow & 0 \\ & & & & & \nwarrow & & & \\ 0 & \longrightarrow & \operatorname{Ker}\alpha & \longrightarrow & \operatorname{Ker}\beta & \longrightarrow & \operatorname{Ker}\gamma & & \end{array}$$

Proof. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be the following cochain complexes:

$$\begin{array}{ll} \mathcal{A} & 0 \longrightarrow A^0 \xrightarrow{\alpha} A^1 \longrightarrow 0 \\ \mathcal{B} & 0 \longrightarrow B^0 \xrightarrow{\beta} B^1 \longrightarrow 0 \\ \mathcal{C} & 0 \longrightarrow C^0 \xrightarrow{\gamma} C^1 \longrightarrow 0 \end{array}$$

So, the given commutative diagram with exact rows is a short exact sequence of cochain complexes:

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

Therefore, by The Zig-Zag Lemma, there is a long exact sequence at the cohomology level:

$$\begin{array}{ccccc} H^{k+1}(\mathcal{A}) & \longrightarrow & \cdots & & \\ H^{k}(\mathcal{A}) & \longrightarrow & H^{k}(\mathcal{B}) & \longrightarrow & H^{k}(\mathcal{C}) \\ & & \cdots & \longrightarrow & H^{k-1}(\mathcal{C}) \end{array} \tag{10.58}$$

Notice that, for $k \neq 0, 1$

$$H^k(\mathcal{A}) = H^k(\mathcal{B}) = H^k(\mathcal{C}) = 0, \tag{10.59}$$

since the cochain groups are trivial for $k \neq 0, 1$. For $k = 0$,

$$H^0(\mathcal{A}) = \frac{\operatorname{Ker}\alpha}{\operatorname{im}(0 \to A^0)} = \operatorname{Ker}\alpha, \tag{10.60}$$

$$H^0(\mathcal{B}) = \frac{\operatorname{Ker}\beta}{\operatorname{im}(0 \to B^0)} = \operatorname{Ker}\beta, \tag{10.61}$$

$$H^0(\mathcal{C}) = \frac{\operatorname{Ker}\gamma}{\operatorname{im}(0 \to C^0)} = \operatorname{Ker}\gamma. \tag{10.62}$$

For $k = 1$,

$$H^1(\mathcal{A}) = \frac{\operatorname{Ker}(A^1 \to 0)}{\operatorname{im}\alpha} = \frac{A^1}{\operatorname{im}\alpha} = \operatorname{cok}\alpha, \tag{10.63}$$

$$H^1(\mathcal{B}) = \frac{\operatorname{Ker}(B^1 \to 0)}{\operatorname{im}\beta} = \frac{B^1}{\operatorname{im}\beta} = \operatorname{cok}\beta, \tag{10.64}$$

$$H^1(\mathcal{C}) = \frac{\operatorname{Ker}(C^1 \to 0)}{\operatorname{im}\gamma} = \frac{A^1}{\operatorname{im}\gamma} = \operatorname{cok}\gamma. \tag{10.65}$$

So (10.58) becomes

$$\begin{array}{ccccccccc} & & \operatorname{cok}\alpha & \longrightarrow & \operatorname{cok}\beta & \longrightarrow & \operatorname{cok}\gamma & \longrightarrow & 0 \longrightarrow \cdots \\ \cdots \longrightarrow & 0 & \longrightarrow \operatorname{Ker}\alpha & \longrightarrow & \operatorname{Ker}\beta & \longrightarrow & \operatorname{Ker}\gamma & & \end{array}$$

where the dots represent trivial vector spaces (containing only the zero vector). ■

§10.4 The Mayer–Vietoris Sequence

Let $\{U, V\}$ be an open cover of a manifold M, and let $i_U : U \hookrightarrow M$, $i_U(p) = p$, be the inclusion map. Then the pullback

$$(i_U^*)_k : \Omega^k(M) \to \Omega^k(U)$$

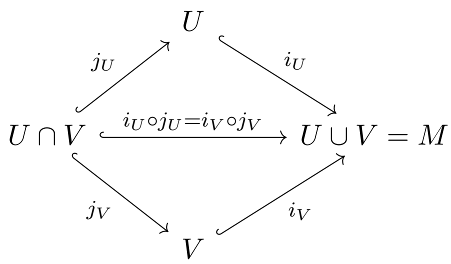
is the restriction map that restricts the domain of a k-form on M to U (Example 4.3). In other words,

$$(i_U^*)_k\,\omega = \omega|_U, \tag{10.66}$$

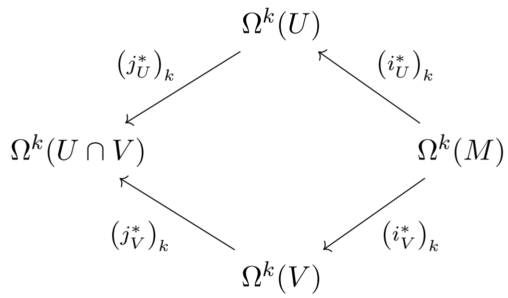
for $\omega \in \Omega^k(M)$. Similarly,

$$(i_V^*)_k \, \omega = \omega|_V, \tag{10.67}$$

In fact, there are 4 relevant inclusion maps forming a commutative diagram:



These inclusions induce the following commutative diagram of vector spaces:



Similarly as (10.67), $(j_U^*)_k$ and $(j_V^*)_k$ also restricts the domain of the smooth k-form. In other words,

$$(j_U^*)_k \, \omega = \omega|_{U \cap V} \text{ and } (j_V^*)_k \, \tau = \tau|_{U \cap V}. \tag{10.68}$$

We then define the following linear maps between vector spaces:

$$i_k : \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V) \text{ and } j_k : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V),$$

defined by

$$i_k(\sigma) = ((i_U^*)_k \, \sigma, (i_V^*)_k \, \sigma) = \left(\sigma|_U, \sigma|_V\right); \tag{10.69}$$

$$j_k(\omega, \tau) = (j_V^*)_k \, \tau - (j_U^*)_k \, \omega = \tau|_{U \cap V} - \omega|_{U \cap V}. \tag{10.70}$$

If $U \cap V$ is empty, then we define $\Omega^k(U \cap V) = 0$, and in this case $j_k : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V)$ is simply the zero map. We call i_k the **restriction map** and j_k the **difference map**. The exterior derivative $\widetilde{d}$ on $\Omega^*(U) \oplus \Omega^*(V)$ is given by

$$\widetilde{\mathrm{d}}(\omega, \tau) = (\mathrm{d}_U \omega, \mathrm{d}_V \tau), \tag{10.71}$$

for $\omega \in \Omega^k(U)$ and $\tau \in \Omega^k(V)$, where d_U and d_V are exterior derivative operators on the open subsets U and V, respectively. We can interpret $\Omega^k(U) \oplus \Omega^k(V)$ as $\Omega^k(U \sqcup V)$, where $U \sqcup V$ is the disjoint union of U and V.

Proposition 10.5

Both the restriction map i_k and the difference map j_k commute with exterior derivatives, i.e. $\{i_k\}$ and $\{j_k\}$ are cochain maps.

Proof. Consider the pullback maps $(i_U^*)_k : \Omega^k(M) \to \Omega^k(U)$ and $(i_V^*)_k : \Omega^k(M) \to \Omega^k(V)$. Since exterior derivative commutes with pullback, the following diagrams commute:

$$\begin{CD} \Omega^k(M) @>{(i_U^*)_k}>> \Omega^k(U) \\ @V{\mathrm{d}}VV @VV{\mathrm{d}_U}V \\ \Omega^{k+1}(M) @>>{(i_U^*)_{k+1}}> \Omega^{k+1}(U) \end{CD} \qquad \begin{CD} \Omega^k(M) @>{(i_V^*)_k}>> \Omega^k(V) \\ @V{\mathrm{d}}VV @VV{\mathrm{d}_V}V \\ \Omega^{k+1}(M) @>>{(i_V^*)_{k+1}}> \Omega^{k+1}(V) \end{CD}$$

In other words,

$$\mathrm{d}_U \circ (i_U^*)_k = (i_U^*)_{k+1} \circ \mathrm{d}, \text{ and } \mathrm{d}_V \circ (i_V^*)_k = (i_V^*)_{k+1} \circ \mathrm{d}. \tag{10.72}$$

Now, for $\sigma \in \Omega^k(M)$,

$$\begin{aligned} \left(\widetilde{\mathrm{d}} \circ i_k\right) \sigma &= \widetilde{\mathrm{d}}\left((i_U^*)_k \sigma, (i_V^*)_k \sigma\right) \\ &= \left(\mathrm{d}_U (i_U^*)_k \sigma, \mathrm{d}_V (i_V^*)_k \sigma\right) \\ &= \left((i_U^*)_{k+1} \mathrm{d}\sigma, (i_V^*)_{k+1} \mathrm{d}\sigma\right) \\ &= (i_{k+1} \circ \mathrm{d})(\sigma). \end{aligned}$$

Therefore,

$$\widetilde{\mathrm{d}} \circ i_k = i_{k+1} \circ \mathrm{d}. \tag{10.73}$$

Again, from the commutativity of pullback with exterior derivative operator, we have the following commutative diagrams:

$$\begin{CD} \Omega^k(U) @>{(j_U^*)_k}>> \Omega^k(U\cap V) \\ @V{\mathrm{d}_U}VV @VV{\mathrm{d}_{U\cap V}}V \\ \Omega^{k+1}(U) @>>{(j_U^*)_{k+1}}> \Omega^{k+1}(U\cap V) \end{CD} \qquad \begin{CD} \Omega^k(V) @>{(j_V^*)_k}>> \Omega^k(U\cap V) \\ @V{\mathrm{d}_V}VV @VV{\mathrm{d}_{U\cap V}}V \\ \Omega^{k+1}(V) @>>{(j_V^*)_{k+1}}> \Omega^{k+1}(U\cap V) \end{CD}$$

In other words,

$$\mathrm{d}_{U\cap V} \circ (j_U^*)_k = (j_U^*)_{k+1} \circ \mathrm{d}_U, \text{ and } \mathrm{d}_{U\cap V} \circ (j_V^*)_k = (j_V^*)_{k+1} \circ \mathrm{d}_V. \tag{10.74}$$

Now, for $(\omega, \tau) \in \Omega^k(U) \oplus \Omega^k(V)$,

$$\begin{aligned} (\mathrm{d}_{U\cap V} \circ j_k)(\omega, \tau) &= \mathrm{d}_{U\cap V}\left((j_V^*)_k \tau - (j_U^*)_k \omega\right) \\ &= \mathrm{d}_{U\cap V} (j_V^*)_k \tau - \mathrm{d}_{U\cap V} (j_U^*)_k \omega \\ &= (j_V^*)_{k+1} \mathrm{d}_V \tau - (j_U^*)_{k+1} \mathrm{d}_U \omega \\ &= j_k (\mathrm{d}_U \omega, \mathrm{d}_V \tau) \\ &= \left(j_k \circ \widetilde{\mathrm{d}}\right)(\omega, \tau). \end{aligned}$$

Therefore,

$$\mathrm{d}_{U\cap V} \circ j_k = j_k \circ \widetilde{\mathrm{d}}. \tag{10.75}$$

■

Proposition 10.6

For each $k \geq 0$, the sequence

$$0 \longrightarrow \Omega^k(M) \xrightarrow{i_k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j_k} \Omega^k(U \cap V) \longrightarrow 0 \tag{10.76}$$

is exact.

Proof. The above sequence can easily be seen to be exact at $\Omega^k(M)$ by noticing that $\operatorname{Ker} i_k = 0$. Indeed, if

$$i_k(\sigma) = \left(\sigma|_U, \sigma|_V\right) = (0,0), \tag{10.77}$$

then $\sigma = 0$ on $U \cup V = M$. Therefore, $\operatorname{Ker} i_k = 0$.

Now we are going to prove that $\operatorname{im} i_k = \operatorname{Ker} j_k$. Let's take $(\omega, \tau) \in \operatorname{Ker} j_k \subseteq \Omega^k(U) \oplus \Omega^k(V)$. Then

$$0 = j_k(\omega, \tau) = \tau|_{U\cap V} - \omega|_{U\cap V}. \tag{10.78}$$

So ω and τ agree on $U \cap V$. So we can define $\sigma \in \Omega^k(M)$ as

$$\sigma_p = \begin{cases} \omega_p & \text{if } p \in U, \\ \tau_p & \text{if } p \in V. \end{cases} \tag{10.79}$$

σ is well-defined since ω and τ agree on $U \cap V$. Then we have

$$i_k(\sigma) = \left(\sigma|_U, \sigma|_V\right) = (\omega, \tau). \tag{10.80}$$

So $(\omega, \tau) \in \operatorname{im} i_k$, proving

$$\operatorname{Ker} j_k \subseteq \operatorname{im} i_k. \tag{10.81}$$

On the other hand, for any $\sigma \in \Omega^k(M)$,

$$j_k(i_k\sigma) = j_k\left(\sigma|_U, \sigma|_V\right) = \left(\sigma|_V\right)|_{U\cap V} - \left(\sigma|_U\right)|_{U\cap V} = \sigma|_{U\cap V} - \sigma|_{U\cap V} = 0. \tag{10.82}$$

Therefore,

$$\operatorname{im} i_k \subseteq \operatorname{Ker} j_k. \tag{10.83}$$

From (10.81) and (10.83), we have

$$\operatorname{im} i_k = \operatorname{Ker} j_k. \tag{10.84}$$

Now we are only left to prove that $j_k : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V)$ is surjective. Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to the open cover $\{U, V\}$ of M. Suppose $\omega \in \Omega^k(U \cap V)$. Then we define $\omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$ as follows:

$$\omega_U = \begin{cases} \rho_V\omega & \text{on } U \cap V, \\ 0 & \text{on } U \setminus (U \cap V). \end{cases} \qquad \omega_V = \begin{cases} \rho_U\omega & \text{on } U \cap V, \\ 0 & \text{on } V \setminus (U \cap V). \end{cases} \tag{10.85}$$

ω_U is called the **extension by zero** of $\rho_V\omega$ from $U \cap V$ to U; and similarly, ω_V is called the extension by zero of $\rho_U\omega$ from $U \cap V$ to V. We now need to show that ω_U and ω_V are smooth.

Clearly, ω_U is smooth on $U \cap V$. Suppose $q \in U\setminus(U \cap V) = U\setminus V$. Since $\operatorname{supp}\rho_V \subseteq V$, $q \in U\setminus\operatorname{supp}\rho_V$. Since $\operatorname{supp}\rho_V$ is closed, $U \setminus \operatorname{supp}\rho_V$ is open. So we can find a coordinate neighborhood (W, φ) about q such that $W \subseteq U \setminus \operatorname{supp}\rho_V$. Now, since W is disjoint from ρ_V, $\omega_U = 0$ on W. Therefore, ω_U is smooth on W. In particular, ω_U is smooth at $q \in U \setminus (U \cap V)$. Since $q \in U \setminus (U \cap V)$ is arbitrary, ω_U is smooth on all of $U \setminus (U \cap V)$. Therefore, ω_U is smooth. Similarly, ω_V is also smooth.

Now, since ω_U and ω_V are smooth, $\omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$. Now,

$$j_k(-\omega_U, \omega_V) = \omega_V|_{U\cap V} + \omega_U|_{U\cap V} = \rho_U\omega + \rho_V\omega = \omega. \tag{10.86}$$

Therefore, j_k is surjective. Hence, (10.76) is a short exact sequence. ■

Lemma 10.7

The k-th cohomology vector space of $U \sqcup V$ is isomorphic to $H^k(U) \oplus H^k(V)$.

Proof. The cochain complex $\Omega^*(U) \oplus \Omega^*(V)$ is

$$\cdots \longrightarrow \Omega^{k-1}(U) \oplus \Omega^{k-1}(V) \xrightarrow{\widetilde{\mathrm{d}}_{k-1}} \Omega^{k}(U) \oplus \Omega^{k}(V) \xrightarrow{\widetilde{\mathrm{d}}_{k}} \Omega^{k+1}(U) \oplus \Omega^{k+1}(V) \longrightarrow \cdots$$

k-th cohomology vector space of the cochain complex $\Omega^*(U) \oplus \Omega^*(V)$ is

$$H^k(U \sqcup V) = \frac{\operatorname{Ker} \widetilde{\mathrm{d}}_k}{\operatorname{im} \widetilde{\mathrm{d}}_{k-1}}. \tag{10.87}$$

For $\omega \in \Omega^k(U)$ and $\tau \in \Omega^k(V)$, $\widetilde{\mathrm{d}}_k(\omega, \tau) = ((\mathrm{d}_U)_k \omega, (\mathrm{d}_V)_k \tau)$. So

$$\begin{aligned} (\omega, \tau) \in \operatorname{Ker} \widetilde{\mathrm{d}}_k &\iff (\mathrm{d}_U)_k \omega = (\mathrm{d}_V)_k \tau = 0 \\ &\iff \omega \in \operatorname{Ker}(\mathrm{d}_U)_k \text{ and } \tau \in \operatorname{Ker}(\mathrm{d}_V)_k \\ &\iff (\omega, \tau) \in \operatorname{Ker}(\mathrm{d}_U)_k \oplus \operatorname{Ker}(\mathrm{d}_V)_k. \end{aligned}$$

Therefore,

$$\operatorname{Ker} \widetilde{\mathrm{d}}_k = \operatorname{Ker}(\mathrm{d}_U)_k \oplus \operatorname{Ker}(\mathrm{d}_V)_k \tag{10.88}$$

Again,

$$\begin{aligned} (\omega, \tau) \in \operatorname{im} \widetilde{\mathrm{d}}_{k-1} &\iff (\omega, \tau) = \widetilde{\mathrm{d}}_{k-1}(\alpha, \beta) \text{ for some } (\alpha, \beta) \in \Omega^{k-1}(U) \oplus \Omega^{k-1}(V) \\ &\iff \omega = (\mathrm{d}_U)_{k-1} \alpha \text{ and } \tau = (\mathrm{d}_V)_{k-1} \beta, \text{ for some } \alpha \in \Omega^{k-1}(U), \ \beta \in \Omega^{k-1}(V) \\ &\iff \omega \in \operatorname{im}(\mathrm{d}_U)_{k-1} \text{ and } \tau \in \operatorname{im}(\mathrm{d}_V)_{k-1} \\ &\iff (\omega, \tau) \in \operatorname{im}(\mathrm{d}_U)_{k-1} \oplus \operatorname{im}(\mathrm{d}_V)_{k-1}. \end{aligned}$$

Therefore,

$$\operatorname{im} \widetilde{\mathrm{d}}_{k-1} = \operatorname{im}(\mathrm{d}_U)_{k-1} \oplus \operatorname{im}(\mathrm{d}_V)_{k-1}. \tag{10.89}$$

As a result,

$$H^k(U \sqcup V) = \frac{\operatorname{Ker} \widetilde{\mathrm{d}}_k}{\operatorname{im} \widetilde{\mathrm{d}}_{k-1}} = \frac{\operatorname{Ker}(\mathrm{d}_U)_k \oplus \operatorname{Ker}(\mathrm{d}_V)_k}{\operatorname{im}(\mathrm{d}_U)_{k-1} \oplus \operatorname{im}(\mathrm{d}_V)_{k-1}}. \tag{10.90}$$

Let us now consider the surjective linear map $\psi : \operatorname{Ker}(\mathrm{d}_U)_k \oplus \operatorname{Ker}(\mathrm{d}_V)_k \to H^k(U) \oplus H^k(V)$ defined by

$$\psi(\omega, \tau) = ([\omega], [\tau]). \tag{10.91}$$

Now,

$$\begin{aligned} (\omega, \tau) \in \operatorname{Ker} \psi &\iff [\omega] = [0] \in H^k(U) \text{ and } [\tau] = [0] \in H^k(V) \\ &\iff \omega = \mathrm{d}_U \alpha \text{ and } \tau = \mathrm{d}_V \beta, \text{ for some } \alpha \in \Omega^{k-1}(U), \ \beta \in \beta \in \Omega^{k-1}(V) \\ &\iff (\omega, \tau) \in \operatorname{im}(\mathrm{d}_U)_{k-1} \oplus \operatorname{im}(\mathrm{d}_V)_{k-1}. \end{aligned}$$

Therefore, $\operatorname{Ker} \psi = \operatorname{im}(\mathrm{d}_U)_{k-1} \oplus \operatorname{im}(\mathrm{d}_V)_{k-1}$. Hence, by the first isomorphism theorem, we have the following isomorphism of vector spaces:

$$H^k(U) \oplus H^k(V) = \operatorname{im} \psi \cong \frac{\operatorname{Ker}(\mathrm{d}_U)_k \oplus \operatorname{Ker}(\mathrm{d}_V)_k}{\operatorname{Ker} \psi} = \frac{\operatorname{Ker}(\mathrm{d}_U)_k \oplus \operatorname{Ker}(\mathrm{d}_V)_k}{\operatorname{im}(\mathrm{d}_U)_{k-1} \oplus \operatorname{im}(\mathrm{d}_V)_{k-1}}. \tag{10.92}$$

So $H^k(U \sqcup V)$ is isomorphic to $H^k(U) \oplus H^k(V)$. ■

Theorem 10.8 (The Mayer–Vietoris Sequence)

Let U and V be open subsets of M such that $U \cup V = M$. Then there is a long exact sequence in cohomology:

$$\cdots \longrightarrow H^k(M) \longrightarrow H^k(U) \oplus H^k(V) \longrightarrow H^k(U \cap V) \longrightarrow H^{k+1}(M) \longrightarrow \cdots$$

called the **Mayer–Vietoris sequence**.

Proof. By Proposition 10.6, we have a short exact sequence of cochain complexes:

$$0 \longrightarrow \Omega^*(M) \xrightarrow{\ i\ } \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{\ j\ } \Omega^*(U \cap V) \longrightarrow 0. \tag{10.93}$$

Then by The Zig-Zag Lemma, (10.93) induces a long exact sequence in cohomology:

$$\cdots \xrightarrow{\left(j^{\#}\right)^{k-1}} H^{k-1}(U\cap V) \xrightarrow{\left(\mathrm{d}^{\#}\right)_{k-1}} H^k(M) \xrightarrow{\left(i^{\#}\right)^{k}} H^k(U) \oplus H^k(V) \xrightarrow{\left(j^{\#}\right)^{k}} H^k(U\cap V) \xrightarrow{\left(\mathrm{d}^{\#}\right)_{k}} H^{k+1}(M) \xrightarrow{\left(i^{\#}\right)^{k+1}} \cdots \tag{10.94}$$

■

In this sequence (10.94), $\left(i^{\#}\right)^k$ and $\left(j^{\#}\right)^k$ are induced from i_k and j_k:

$$\left(i^{\#}\right)^k [\sigma] = [i_k(\sigma)] = \left(\left[\sigma|_U\right], \left[\sigma|_V\right]\right), \tag{10.95}$$

$$\left(j^{\#}\right)^k ([\omega], [\tau]) = [j_k(\omega, \tau)] = \left[\tau|_{U\cap V} - \omega|_{U\cap V}\right]. \tag{10.96}$$

The connecting homomorphism $\left(\mathrm{d}^{\#}\right)^k : H^k(U\cap V) \to H^{k+1}(M)$ is cooked up in 3 steps using the same recipe as we did in § 10.3.

$$\begin{array}{ccccc}
\Omega^{k+1}(M) & \xrightarrow{i_{k+1}} & \Omega^{k+1}(U)\oplus\Omega^{k+1}(V) & & \\
 & & \uparrow{\scriptstyle \widetilde{\mathrm{d}}} & & \\
 & & \Omega^{k}(U)\oplus\Omega^{k}(V) & \xrightarrow{j_k} & \Omega^{k}(U\cap V)
\end{array}$$

$$\begin{array}{ccccc}
\alpha & \xrightarrow{i_{k+1}} & (-\mathrm{d}_U\xi_U, \mathrm{d}_V\xi_V) & \xrightarrow{j_{k+1}} & 0 \\
 & & \uparrow{\scriptstyle \widetilde{\mathrm{d}}} & & \uparrow{\scriptstyle \mathrm{d}_{U\cap V}} \\
 & & (-\xi_U, \xi_V) & \xrightarrow{j_k} & \xi
\end{array}$$

(i) We start with a closed k-form $\xi \in \Omega^k(U\cap V)$ and using a partition of unity $\{\rho_U, \rho_V\}$ subordinate to the open cover $\{U, V\}$, one can extend $\rho_U\xi$ by zero from $U\cap V$ to a k-form ξ_V on V, and extend $\rho_V\xi$ by zero from $U\cap V$ to a k-form ξ_U on U. Then

$$j_k(-\xi_U, \xi_V) = \xi_V|_{U\cap V} + \xi_U|_{U\cap V} = \rho_U\xi + \rho_V\xi = \xi. \tag{10.97}$$

(ii) Since $\{j_k\}$ is a cochain map, the following square commutes:

$$\begin{array}{ccc}
\Omega^{k+1}(U)\oplus\Omega^{k+1}(V) & \xrightarrow{j_{k+1}} & \Omega^{k+1}(U\cap V) \\
\uparrow{\scriptstyle \widetilde{\mathrm{d}}} & & \uparrow{\scriptstyle \mathrm{d}_{U\cap V}} \\
\Omega^{k}(U)\oplus\Omega^{k}(V) & \xrightarrow{j_k} & \Omega^{k}(U\cap V)
\end{array}$$

Hence,

$$\begin{aligned} j_{k+1}\left(-\mathrm{d}_U\xi_U, \mathrm{d}_V\xi_V\right) &= \left(j_{k+1}\circ \widetilde{\mathrm{d}}\right)\left(-\xi_U, \xi_V\right) = \left(\mathrm{d}_{U\cap V}\circ j_k\right)\left(-\xi_U, \xi_V\right) \\ &= \mathrm{d}_{U\cap V}\xi = 0. \end{aligned}$$

(iii) So $\left(-\mathrm{d}_U\xi_U, \mathrm{d}_V\xi_V\right) \in \operatorname{Ker} j_{k+1} = \operatorname{im} i_{k+1}$. Therefore, $-\mathrm{d}_U\xi_U$ on U and $\mathrm{d}_V\xi_V$ on V patch together to yield a global $(k+1)$-form $\alpha \in \Omega^{k+1}(M)$ such that

$$i_{k+1}(\alpha) = \left(-\mathrm{d}_U\xi_U, \mathrm{d}_V\xi_V\right). \tag{10.98}$$

Since $\{i_k\}$ is a chain map, the following square commutes:

$$\begin{array}{ccc} \Omega^{k+2}(M) & \xrightarrow{i_{k+2}} & \Omega^{k+2}(U)\oplus\Omega^{k+2}(V) \\ \uparrow \mathrm{d} & & \uparrow \widetilde{\mathrm{d}} \\ \Omega^{k+1}(M) & \xrightarrow[i_{k+1}]{} & \Omega^{k+1}(U)\oplus\Omega^{k+1}(V) \end{array}$$

So we have

$$i_{k+2}(\mathrm{d}\alpha) = \widetilde{\mathrm{d}}\left(i_{k+1}\alpha\right) = \widetilde{\mathrm{d}}\left(-\mathrm{d}_U\xi_U, \mathrm{d}_V\xi_V\right) = (0,0).$$

Since i_{k+2} is injective, $\mathrm{d}\alpha = 0$, i.e. $\alpha \in \Omega^{k+1}(M)$ is also a closed form. So we define

$$\left(\mathrm{d}^{\#}\right)^k [\xi] = [\alpha] \in H^{k+1}(M). \tag{10.99}$$

Since $\Omega^k(M) = 0$ for $k < 0$, The Mayer–Vietoris Sequence starts with

$$0 \longrightarrow H^0(M) \longrightarrow H^0(U)\oplus H^0(V) \longrightarrow H^0(U\cap V) \longrightarrow H^1(M) \longrightarrow \cdots$$

Proposition 10.9

In The Mayer–Vietoris Sequence, if U, V, and $U\cap V$ are connected and nonempty, then

(i) M is connected, and

$$0 \longrightarrow H^0(M) \xrightarrow{\left(i^{\#}\right)^0} H^0(U)\oplus H^0(V) \xrightarrow{\left(j^{\#}\right)^0} H^0(U\cap V) \longrightarrow 0$$

is exact;

(ii) we may start The Mayer–Vietoris Sequence with

$$0 \longrightarrow H^1(M) \xrightarrow{\left(i^{\#}\right)^1} H^1(U)\oplus H^1(V) \xrightarrow{\left(j^{\#}\right)^1} H^1(U\cap V) \xrightarrow{\left(\mathrm{d}^{\#}\right)^1} H^2(M) \longrightarrow \cdots$$

Proof. (i) The connectedness of M follows from the connectedness of U and V and that U and V are not disjoint using point set topological argument. But let us try to deduce it using The Mayer–Vietoris Sequence.

On a nonempty connected open set, the de Rham cohomology in dimension 0 is simply the vector space of constant functions (Proposition 9.1). The constant functions are characterized by real numbers. Additionally, if $u \in \mathbb{R}$ represents a constant function on U, then on $U\cap V$, it is the same constant function u, i.e. $u|_{U\cap V} = u$[1]. Therefore, the map $\left(j^{\#}\right)^0 : H^0(U)\oplus H^0(V) \to H^0(U\cap V)$ is given by

$$\left(j^{\#}\right)^0(u,v) = v|_{U\cap V} - u|_{U\cap V} = v - u. \tag{10.100}$$

Clearly, $\left(j^{\#}\right)^0$ is surjective.

[1]Here we are abusing the notation by denoting the constant function and its value, which is a real number, by the same symbol u.

$$0 \longrightarrow H^0(M) \xrightarrow{\left(i^{\#}\right)^0} H^0(U) \oplus H^0(V) \xrightarrow{\left(j^{\#}\right)^0} H^0(U \cap V) \xrightarrow{\left(\mathrm{d}^{\#}\right)^0} H^1(M) \longrightarrow \cdots$$

Surjectivity of $\left(j^{\#}\right)^0$ implies that $\mathrm{im}\left(j^{\#}\right)^0 = H^0(U \cap V)$. Exactness of the Mayer–Vietoris sequence above implies

$$\mathrm{Ker}\left(\mathrm{d}^{\#}\right)^0 = \mathrm{im}\left(j^{\#}\right)^0 = H^0(U \cap V). \tag{10.101}$$

So $\left(\mathrm{d}^{\#}\right)^0 : H^0(U \cap V) \to H^1(M)$ is the zero map. Thusm the Mayer–Vietoris sequence starts with

$$0 \longrightarrow H^0(M) \xrightarrow{\left(i^{\#}\right)^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\left(j^{\#}\right)^0} \mathbb{R} \xrightarrow{\left(\mathrm{d}^{\#}\right)^0} 0 \tag{10.102}$$

The above sequence is short exact, sice the Mayer–Vietoris sequence is exact. Exactness at $H^0(M)$ implies $\mathrm{Ker}\left(i^{\#}\right)^0 = 0$, i.e. $\left(i^{\#}\right)^0$ is injective. Therefore, by the first isomorphism theorem,

$$\mathrm{im}\left(i^{\#}\right)^0 \cong \frac{H^0(M)}{\mathrm{Ker}\,(i^{\#})^0} = H^0(M). \tag{10.103}$$

Exactness of (10.102) at $\mathbb{R} \oplus \mathbb{R}$ implies

$$\mathrm{im}\left(i^{\#}\right)^0 = \mathrm{Ker}\left(j^{\#}\right)^0. \tag{10.104}$$

$\left(j^{\#}\right)^0 (u, v)\,(u, v) = v - u$, so

$$\mathrm{Ker}\left(j^{\#}\right)^0 = \{(u, v) \in \mathbb{R} \oplus \mathbb{R} \mid v - u = 0\} = \{(u, u) \in \mathbb{R} \oplus \mathbb{R}\} \cong \mathbb{R}. \tag{10.105}$$

Therefore, combining (10.103), (10.104) and (10.105), we get

$$H^0(M) \cong \mathrm{im}\,\mathrm{im}\left(i^{\#}\right)^0 = \mathrm{Ker}\left(j^{\#}\right)^0 \cong \mathbb{R}. \tag{10.106}$$

So $H^0(M) \cong \mathbb{R}$, i.e. M is connected.

(ii) We have deduced earlier that $\left(\mathrm{d}^{\#}\right)^0 : H^0(U \cap V) \to H^1(M)$ is the zero map. Thus, in the Mayer–Vietoris sequence, the sequence of the following two maps

$$H^0(U \cap V) \xrightarrow{\left(\mathrm{d}^{\#}\right)^0} H^1(M) \xrightarrow{\left(i^{\#}\right)^1} H^1(U) \oplus H^1(V)$$

may be replaced by

$$0 \longrightarrow H^1(M) \xrightarrow{\left(i^{\#}\right)^1} H^1(U) \oplus H^1(V)$$

without affecting exactness. In other words, no information of the Mayer–Vietoris sequence is lost if we have

$$0 \longrightarrow H^0(M) \xrightarrow{\left(i^{\#}\right)^0} H^0(U) \oplus H^0(V) \xrightarrow{\left(j^{\#}\right)^0} H^0(U \cap V) \longrightarrow 0$$

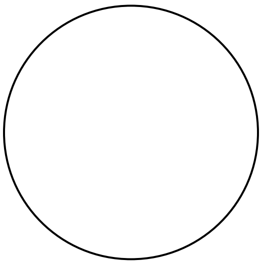
to be short exact, and we have a long exact sequence as follows:

$$0 \longrightarrow H^1(M) \xrightarrow{\left(i^{\#}\right)^1} H^1(U) \oplus H^1(V) \xrightarrow{\left(j^{\#}\right)^1} H^1(U \cap V) \xrightarrow{\left(\mathrm{d}^{\#}\right)^1} H^2(M) \longrightarrow \cdots$$

■

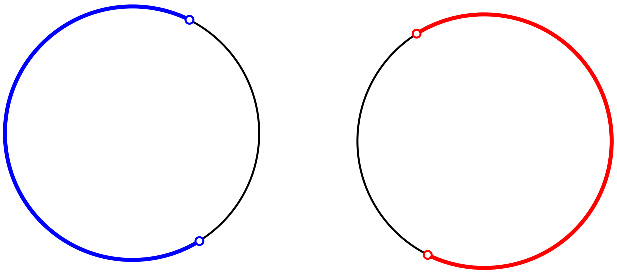
Example 10.3 (Cohomology of circle using Mayer–Vietoris sequence)**.** Let S^1 be the circle in $\mathbb{R}^2$,

$$S^1 = \left\{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| = 1\right\}.$$



S^1

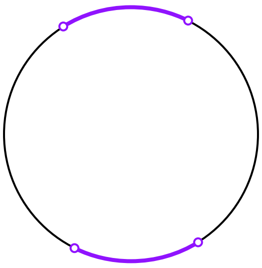
Since S^1 is connected, by Proposition 9.1, $H^0(S^1) = \mathbb{R}$. Since S^1 is a one-dimensional manifold, by Proposition 9.2, $H^k(S^1) = 0$ for $k \geq 2$. Now we want to compute $H^1(S^1)$ using The Mayer–Vietoris Sequence. We cover S^1 by open sets U and V as follows:



U V

U and V are open arcs on the circle S^1. Their intersection is the following (disjoint) union of two open arcs A and B:

A



B

Open arcs are diffeomorphic to open intervals on $\mathbb{R}$, which are diffeomorphic to the whole $\mathbb{R}$. Therefore, by diffeomorphism invariance,

$$H^1(U) \cong H^1(V) \cong H^1(A) \cong H^1(B) \cong H^1(\mathbb{R}) = 0. \tag{10.107}$$

As a result,

$$H^1(U \cap V) = H^1(A \sqcup B) \cong H^1(A) \oplus H^1(B) = 0. \tag{10.108}$$

Furthermore,

$$H^1(U) \oplus H^1(V) = 0. \tag{10.109}$$

In dimension 0, since open arcs are diffeomorphic to $\mathbb{R}$,

$$H^0(U) \cong H^0(V) \cong H^0(A) \cong H^0(B) \cong H^0(\mathbb{R}) = \mathbb{R}. \tag{10.110}$$

As a result,

$$H^0(U \cap V) = H^0(A \sqcup B) \cong H^0(A) \oplus H^0(B) = \mathbb{R} \oplus \mathbb{R}. \tag{10.111}$$

Furthermore,

$$H^0(U) \oplus H^1(V) = \mathbb{R} \oplus \mathbb{R}. \tag{10.112}$$

So, the Mayer–Vietoris sequence

$$\begin{array}{ccccccc} & & H^1\left(S^1\right) & \xrightarrow{\left(i^\#\right)^1} & H^1\left(U\right)\oplus H^1\left(V\right) & \longrightarrow & \cdots \\ & & & & \uparrow{\scriptstyle \left(\mathrm{d}^\#\right)^0} & & \\ 0 & \longrightarrow & H^0\left(S^1\right) & \xrightarrow[\left(i^\#\right)^0]{} & H^0\left(U\right)\oplus H^0\left(V\right) & \xrightarrow[\left(j^\#\right)^0]{} & H^0\left(U\cap V\right) \end{array}$$

becomes

$$0 \longrightarrow \mathbb{R} \xrightarrow{\left(i^\#\right)^0} \mathbb{R}\oplus\mathbb{R} \xrightarrow{\left(j^\#\right)^0} \mathbb{R}\oplus\mathbb{R} \xrightarrow{\left(\mathrm{d}^\#\right)^0} H^1\left(S^1\right) \longrightarrow 0. \tag{10.113}$$

By the rank-nullity theorem, if $f : V_1 \to V_2$ is a linear map between finite dimensional vector spaces, then

$$\dim V_1 = \operatorname{rank} f + \operatorname{nullity} f = \dim \operatorname{im} f + \dim \operatorname{Ker} f. \tag{10.114}$$

Since (10.113) is exact, $\left(i^\#\right)^0$ is injective. So $\dim \operatorname{Ker}\left(i^\#\right)^0 = 0$. Hence, $\dim \operatorname{im}\left(i^\#\right)^0 = \dim \mathbb{R} = 1$. As a result,

$$\dim \operatorname{Ker}\left(j^\#\right)^0 = \dim \operatorname{im}\left(i^\#\right)^0 = 1. \tag{10.115}$$

So we have

$$\dim \operatorname{im}\left(j^\#\right)^0 = \dim\left(\mathbb{R}\oplus\mathbb{R}\right) - \dim \operatorname{Ker}\left(j^\#\right)^0 = 1. \tag{10.116}$$

Since $\operatorname{Ker}\left(\mathrm{d}^\#\right)^0 = \operatorname{im}\left(j^\#\right)^0$, we have $\dim \operatorname{Ker}\left(\mathrm{d}^\#\right)^0 = 1$. By the exactness of (10.113), $\left(\mathrm{d}^\#\right)^0$ is surjective. Hence,

$$\dim H^1\left(S^1\right) = \dim \operatorname{im}\left(\mathrm{d}^\#\right)^0 = \dim\left(\mathbb{R}\oplus\mathbb{R}\right) - \dim \operatorname{Ker}\left(\mathrm{d}^\#\right)^0 = 1. \tag{10.117}$$

So $H^1\left(S^1\right) \cong \mathbb{R}$. Therefore,

$$H^k(S^1) \cong \begin{cases} \mathbb{R} & \text{for } k = 0, 1 \\ 0 & \text{for } k \geq 2. \end{cases} \tag{10.118}$$