BRAC UNIVERSITY
Inspiring Excellence

MAT314 - Complex Analysis

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0 Complex Analysis Is Cool!

We know a lot about $\mathbb{R}$, the set of real numbers. It has some very cool structures. For example, it has an algebraic structure called *Field*. That means, we have an addition operation $(+)$ and a multiplication operation $(\cdot)$, and they have some really nice properties. They are:

(a) **Associativity of addition and multiplication:** For every $a, b, c \in \mathbb{R}$

$$a + (b + c) = (a + b) + c \text{ and } a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

(b) **Commutativity of addition and multiplication:** For every $a, b \in \mathbb{R}$

$$a + b = b + a \text{ and } a \cdot b = b \cdot a$$

(c) **Additive and multiplicative identity:** There exist two different elements 0 and 1 in $\mathbb{R}$ such that for every $a \in \mathbb{R}$

$$a + 0 = a \text{ and } a \cdot 1 = a$$

(d) **Additive inverses:** For every $a \in \mathbb{R}$, there exists an element in $\mathbb{R}$ — denoted by $-a$ and called the additive inverse of a — such that

$$a + (-a) = 0$$

(e) **Multiplicative inverses:** For every $a \neq 0 \in \mathbb{R}$, there exists an element in $\mathbb{R}$ — denoted by a^{-1} and called the multiplicative inverse of a — such that

$$a \cdot \left(a^{-1}\right) = 1$$

(f) **Distributivity of multiplication over addition:** For every $a, b, c \in \mathbb{R}$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

$\mathbb{R}$ also has a geometric structure called *Metric* or *Distance*. Topologically speaking, it has *Smooth Manifold*[1] structure. For these rich structures, we can do Geometry and Analysis on $\mathbb{R}$. That's all about Real Analysis.

§0.1 $\mathbb{C}$ is discovered or invented?

Real numbers were fine and people were happy. But then some madlad introduced some crazy number i, whose square is -1. Histroically speaking, the "invention" (or discovery?) of i is related with solutions of cubic equation. If $x^3 + px + q = 0$ is a cubic equation, then its real root is given by

$$\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

[1] Ignore if you don't know what that means.

We know that an odd-degree polynomial must have at least one real root. So even if we have $\frac{q^2}{4} + \frac{p^3}{27} < 0$, the quantity stated above should yield some real number. That's how square root of negative numbers were discovered.
Introducing this little "imaginary" number lead to some seemingly crazy (and beautiful) things. The space where all these crazy things happen is the set of complex numbers, denoted by $\mathbb{C}$. In our high school days, $\mathbb{C}$ was introduced as

$$\mathbb{C} = \{a + ib \ : \ a, b \in \mathbb{R}\}$$

But actually you can define $\mathbb{C}$ in several different ways (depending on your purpose). Here we shall use the notion that $\mathbb{C}$ is a 2-dimensional real vector space. That is $\mathbb{C}$ is

Source: https://twitter.com/9gag/status/1051322203533430784

isomorphic to $\mathbb{R}^2$.

$$\mathbb{C} \cong \mathbb{R}^2 \quad \text{as vector space}$$

If we use the vector space notion, then we can define addition and scalar multiplication on $\mathbb{C}$. But vector spaces need not have multiplication. So we need a stronger structure. Turns out we can actually make $\mathbb{C}$ a Field. Not just any ordinary field, $\mathbb{C}$ is an *Algebraic Field*. Meaning, any polynomial with coefficients in $\mathbb{C}$ must have all its roots in $\mathbb{C}$. But it lacks some features. We know that $\mathbb{R}$ is an ordered field. That is

(a) For every $a, b \in \mathbb{R}$, exactly one of the following is true: $a > b$ or $b < a$ or $a = b$

(b) If $a < b$, then for every c we have $a + c < b + c$

(c) If $0 < a$, and $0 < b$ then $0 < ab$

But $\mathbb{C}$ is not an ordered field.
Let's go back to $\mathbb{R}^2$. Suppose we defined multiplication on it, and it's a field now. We wish to show that, there must exist some member $z \in \mathbb{R}^2$ such that $z^2 = -1$.

Theorem 0.1.1
Suppose we defined multiplication on $\mathbb{R}^2$, and it's a field now. Then there must exist some member $z \in \mathbb{R}^2$ such that $z^2 = -1$.

Proof. We shall construct such z. As $\mathbb{R}^2$ is a 2-dimensional vector space, it is spanned by a basis set with 2 elements. Let the basis set be $\{\mathbf{1}, \mathbf{e}\}$. Here $\mathbf{1}$ basically means the pair $(1, 0)$.

Take $z \in \mathbb{R}^2$ such that it's not on the x-axis. It's an element of the vector space, so it can be written as a linear combination of the bases. That is,

$$z = x \cdot \mathbf{1} + y \cdot \mathbf{e} \quad \text{where } x, y \in \mathbb{R} \text{ and } y \neq 0$$

Then we can calculate z^2, using the identity $(p+q)^2 = p^2 + q^2 + 2pq$.

$$z^2 = (x \cdot \mathbf{1} + y \cdot \mathbf{e})^2 = x^2 \cdot \mathbf{1} + y^2 \cdot \mathbf{e}^2 + 2xy \cdot \mathbf{e}$$

$\mathbf{e}^2 \in \mathbb{R}^2$, so it can be written as a linear combination of the bases. Let $\mathbf{e}^2 = a \cdot \mathbf{1} + b \cdot \mathbf{e}$. Plugging this, we get

$$\begin{aligned} z^2 &= x^2 \cdot \mathbf{1} + y^2 \cdot \mathbf{e}^2 + 2xy \cdot \mathbf{e} \\ &= x^2 \cdot \mathbf{1} + ay^2 \cdot \mathbf{1} + by^2 \cdot \mathbf{e} + 2xy \cdot \mathbf{e} \\ &= \left(x^2 + ay^2\right) \cdot \mathbf{1} + \left(by^2 + 2xy\right) \cdot \mathbf{e} \end{aligned}$$

Now, we choose x such that $by^2 + 2xy$ becomes 0. In other words, we choose $x = \frac{-by}{2}$ (we shall fix y later). So we have,

$$z^2 = \left(\left(\frac{-by}{2}\right)^2 + ay^2\right) \cdot \mathbf{1} + 0 \cdot \mathbf{e} = \left(a + \frac{b^2}{4}\right) y^2 \cdot \mathbf{1}$$

Claim — $a + \frac{b^2}{4} < 0$.

Proof. Assume for the sake of contradiction that $a + \frac{b^2}{4} \geq 0$. Then we have a notion of square-root of non-negative real numbers. So let $c = \sqrt{a + \frac{b^2}{4}}$. Now we have,

$$\begin{aligned} z^2 = c^2y^2 \cdot \mathbf{1} &\implies z^2 - c^2y^2 \cdot \mathbf{1}^2 = \mathbf{0} \\ &\implies (z - cy \cdot \mathbf{1})(z + cy \cdot \mathbf{1}) = \mathbf{0} \\ &\implies z - cy \cdot \mathbf{1} = \mathbf{0} \quad \text{or} \quad z + cy \cdot \mathbf{1} = \mathbf{0} \\ &\implies z = cy \cdot \mathbf{1} \quad \text{or} \quad z = -cy \cdot \mathbf{1} \end{aligned}$$

They both contradict the assumption that z does not lie on the x-axis. □

So we have $-\left(a + \frac{b^2}{4}\right) > 0$. Let $c = \sqrt{-\left(a + \frac{b^2}{4}\right)}$. Taking $y = \frac{1}{c}$, we get

$$z^2 = -c^2 \frac{1}{c^2} = -1$$

as desired ■

So if we really wish to give $\mathbb{R}^2$ a field structure, then we must have some i in our space such that $i^2 = -1$. However, apart from $\mathbb{R}^2$ no other Euclidean space $\mathbb{R}^n$ can be given field structure (yeah this had been proven). For higher dimensions, the best we can have is a skew field on $\mathbb{R}^4$. This is called the space of Quarternions. Other than commutativity, it has all other properties of a Field. Also on $\mathbb{R}^8$, you can construct Octonions, but they won't have commutativity and associativity.

§0.2 Some cool examples

Now that we have constructed the field of $\mathbb{C}$, let's witness some games that are being played on this field.

Example 0.2.1

Consider this $\mathbb{R} \to \mathbb{R}$ function:

$$f = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

This function is smooth (meaning it's infinitely differentiable, or C^∞), but this is not analytic. Because $f^{(n)} = 0$ for every n. So the Taylor series about 0 gives us

$$f(0) + \sum_{n=1}^{\infty} \frac{f^n(0)}{n!} x^n = 0 \neq f(x)$$

But this is not the case for functions with complex variables. Every smooth functions on $\mathbb{C}$ is also analytic (Taylor expandable).

Example 0.2.2

Consider this $\mathbb{R} \to \mathbb{R}$ function:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

This function is once-differentiable on $\mathbb{R}$, but the second derivative does not exist. However, for functions with complex variables, if a function is once-differentiable then it is infinitely differentiable.

Source: Mathematical Mathematics Memes

Example 0.2.3

Let f be the Weierstrass function. This function is continuous everywhere, but differentiable nowhere. We construct the following function:

$$F(x) = \int_0^x f(t)\,\mathrm{d}t$$

By the *Fundamental Theorem of Calculas*, $F' = f$. But $F'' = f'$ does not exist. However, such pathological functions does not exist in complex plane. Complex differentiability is kinda like motivational speakers saying, *"If you can do it once, you can do it again."* (and hence recursively infinitely many times)

In complex analysis, C^1 implies C^∞, and C^1 implies analyticity. Hence differentiable, smooth, analytic mean the same thing in complex analysis. If you're tired of finding counterexamples in a real analysis class, complex analysis might be the place for you ;)
.

Example 0.2.4

Suppose we have a smooth function $f : [0, 1] \to \mathbb{R}$, and you are given the value of $f(0)$ and $f(1)$. Just from this little piece of information, can you find out the value of $f(x)$ for every $x \in (0, 1)$? If you did real analysis before, this question is likely to sound nonsense. In real analysis sense, you can't figure out f just from two boundary values.
But in complex analysis, you can use Cauchy integral formula you can determine f just from this little information (with some mild assumptions) :0

Example 0.2.5

If I ask you to give me an example of bounded differentiable function $\mathbb{R} \to \mathbb{R}$, you have many options. Among all other nontrivial answers, $f(x) = c$ seems to be a naive choice, doesn't it?
Well, Liouville's theorem says that on complex plane, the only bunded differentiable function is everywhere constant function :0

Example 0.2.6

Consider a function $f : \mathbb{R} \to \mathbb{R}$ that is non-constant and differentiable on $\mathbb{R}$. An example might be

$$f(x) = \sin(x)$$

In this case, the range of f is $[-1, 1]$. This is just a tiny piece of $\mathbb{R}$. In many cases, the range of non-constant differentiable function is just a subset of $\mathbb{R}$, not the whole set $\mathbb{R}$.
But for non-constant differentiable functions on $\mathbb{C}$, the range is either the whole $\mathbb{C}$, or $\mathbb{C}$ excluding a singleton set. This result is known as Picard's theorem. In particular, one cannot construct a non-constant differentiable function f on $\mathbb{C}$ such that

$$\text{range}(f) = \mathbb{C} \setminus \{p, q\} \quad \text{where } p \neq q$$

Example 0.2.7

Suppose you have an ugly looking set U. Despite being ugly, it's a non-empty simply connected[a] open subset of the complex number plane. If you want to do some geometry/analysis on this set, it's gonna be super hard.
But there is a cool theorem in complex analysis, namely Riemann mapping theorem. This theorem suggests that there exists a biholomorphoc (bijective continuous map, whose inverse is also continuous) map from U to the unit disk $D := \{z \in \mathbb{C} \ : \ |x| < 1\}$.
Using this thoerem, one can do geometry/analysis on the unit disk D and then convert the results back into the ugly looking set U.

[a]If you don't know what that means, you can just ignore for now.

Example 0.2.8

Maybe the most famous function of complex vairables is the Riemann Zeta function $\zeta : \mathbb{C} \to \mathbb{C}$. It is defined as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ when } \operatorname{Re}(s) > 1$$

Note that, this function is not defined on the whole $\mathbb{C}$, it's defined on a subset of $\mathbb{C}$. This function has some really cool properties. One of them is it's holomorphic (complex differentiable). It follows from the fact that, "If a sequence $(f_n)_{n\in\mathbb{N}}$ of holomorphic funnctions uniformly converge to f, then f is also holomorphic." (but this is, in general, not true in real analysis) So we choose f_n as follows:

$$f_n(s) = \sum_{k=1}^{n} \frac{1}{k^s} \text{ and } f_n \to \zeta \text{ uniformly}$$

The most interesting fact about this function is probably its roots. There is an unsolved problem named Riemann hypothesis. The hypothesis suggests that, all the non-trivial roots of ζ can be written as $\frac{1}{2} + ib$. That is, the real part of any non-trivial root is just $\frac{1}{2}$. This hypothesis gives a lot of informations about the density of prime numbers. Many works in modern analytical number theory have been done assuming Riemann hypothesis is true.

However, Riemann zeta function along with the Basel problem gives a very nice proof of the infinitude of primes.

Theorem 0.2.1

There are infinitely many prime numbers.

Proof. Suppose $\mathbb{P}$ denotes the set of prime numbers. Assume for the sake of contradiction that $\#(\mathbb{P}) = k < \infty$. Let n be a positive integer. Then it can be written as prime power factorization as follows:

$$n = p_1^{e_1} p_2^{e^2} \cdots p_k^{e_k} = \prod p_i^{e_i}$$

where $p_i \in \mathbb{P}$ and $e_i \in \mathbb{Z}_{\geq 0}$. Then $\frac{1}{n^s}$ can be written as $\frac{1}{n^s} = \prod \frac{1}{p_i^{e_i s}}$. This can be found in the quantity

$$\left(1+\frac{1}{p_1^s}+\frac{1}{p_1^{2s}}+\cdots\right)\left(1+\frac{1}{p_2^s}+\frac{1}{p_2^{2s}}+\cdots\right)\cdots\left(1+\frac{1}{p_k^s}+\frac{1}{p_k^{2s}}+\cdots\right)$$

Also, this quantity contains every single $\frac{1}{n^s}$, because all natural numbers are composed of primes $p_1, p_2, \ldots, p_k$. Using this, we can rewrite the ζ-function.

$$\begin{aligned}
\zeta(s) &= 1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^s}\\
&= \left(1+\frac{1}{p_1^s}+\frac{1}{p_1^{2s}}+\cdots\right)\left(1+\frac{1}{p_2^s}+\frac{1}{p_2^{2s}}+\cdots\right)\cdots\left(1+\frac{1}{p_k^s}+\frac{1}{p_k^{2s}}+\cdots\right)\\
&= \prod_{p\in\mathbb{P}}\left(1+p^{-s}+p^{-2s}+\cdots\right)\\
&= \prod_{p\in\mathbb{P}}\frac{1}{1-p^{-s}}\\
\implies \zeta(2) &= \prod_{p\in\mathbb{P}}\frac{1}{1-p^{-2}}
\end{aligned}$$

This is a finite product of rational numbers, hence rational. But the Basel problem suggests us that $\zeta(2)=\frac{\pi^2}{6}$, which is irrational. Contradiction! ■

1 Metric Space Crashcourse

There is a quote by Grant Sanderson that says, *"Abstractness is the price of generality."* I think this sums up pure mathematics in a nice way. Mathematicians tend to generalize things so much that at some point things start to become abstract. Metrics are a good example for this. At first, we have the notion of distance on $\mathbb{R}$. Using that and Pythagorian theorem, we can get the distance for $\mathbb{R}^2$. Then it can be generalized for any Euclidean space $\mathbb{R}^n$.
But what about some random set? A random set need not be consisting of elements from $\mathbb{R}^n$. So we don't can't calculate the distance between two elements there. That's why we need to generalize the notion of distance. The generalized distance is called *Metric*.

§1.1 Metric - The Generalized Distance

Now, what properties a distance function should have? Or what properties of a distance function do we want? It should take two elements as input, and one real number as output. From our intuition, the output should always be positive if we calculate the distance of two different elements. Also, the distance should not depend on the order of the inputs. The final thing we want in a distance function is triangle inequality. We shall call any function obeying these properties a metric. Let's look at the formal definition.

> **Definition 1.1.1** (Metric). Let X be a non-empty set. Then a *metric* on X is a map $d : X \times X \to \mathbb{R}$ which satisfies the following:
>
> (i) (Non-negativity) For every $p, q \in X$, $d(p,q) \geq 0$; $d(p,q) = 0$ iff $p = q$.
>
> (ii) (Symmetry) For every $p, q \in X$, $d(p,q) = d(q,p)$.
>
> (iii) (Triangle inequality) For every $p, q, r \in X$, $d(p,q) + d(q,r) \geq d(p,r)$.

Oftentimes we encounter some distance functions d that obey all the properties of metric, except they might give us $d(p,q) = 0$ even if $p \neq q$. Such functions are called *pseudo-metric*. Alright, let's see some examples of metrics.

Example 1.1.1

On $\mathbb{R}$, we define $d(x,y) = |x - y|$. This satisfies non-negativity, symmetry and triangle inequality. So this is a metric.
Similarly, we can define a metric on $\mathbb{R}^n$ using the standard Euclidean norm.

$$d(\mathbf{x}, \mathbf{y}) := \|x - y\| = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$$

It's easy to check that this has the desired properties of a metric.

Definition 1.1.2 (Metric Space)**.** A *metric space* (X, d) consists of a non-empty set X together with a metric d on X.

We shall often say, "X is a metric space" in short. But there is always a metric d associated with X, which we recall when necessary.

§1.2 Open and Closed sets

We can extend the notion of open and closed sets to any metric spaces. For that, we need the definitions of Open and Closed balls first.

Definition 1.2.1 (Open Balls and Closed Balls)**.** Let M be a metric space, $x \in M$, $r > 0$.

$$\text{Open ball of radius } r \text{ around } x \text{ is } B_r(x) := \{y \in M \,:\, d(x, y) < r\}$$
$$\text{Closed ball of radius } r \text{ around } x \text{ is } \overline{B_r}(x) := \{y \in M \,:\, d(x, y) \leq r\}$$

Now we can define open sets.

Definition 1.2.2 (Open Sets)**.** Let M be a metric space. We shall call a subset $S \subseteq M$ open if for every point $x \in S$, there is some $r > 0$ such that $B_r(x) \subseteq S$.

Example 1.2.1

If you look at $\mathbb{R}$, all open intervals are open. If you take any union of them, the resulting set will be an open set.
In fact, there is a famous lemma that, any open set in $\mathbb{R}$ can be expressed as a countable union of disjoint open intervals.

Topologically, closed sets are defined to be the complement of open sets. But there are some alternate equivalent definitions. We shall define closed sets by *limit points*.

Definition 1.2.3 (Limit Points)**.** Let X be a metric space and $E \subseteq X$ be a nonempty subset of X. Then $p \in X$ is a *limit point* of E if for every $\epsilon > 0$, $B_\epsilon(p)$ contains at least one point of E (other than p).

Example 1.2.2

Any real number is a limit point of $\mathbb{Q}$. Same goes for $\mathbb{R} \setminus \mathbb{Q}$. Because if we take any open interval centered at a real number, it is bound to contain some other rational and irrational numbers, no matter how small the interval is.

Example 1.2.3

$\mathbb{Z}$ and $\mathbb{N}$ do not have any limit points in $\mathbb{R}$. For any integer n, if we take $0 < \epsilon < 1$ then $B_\epsilon(n)$ does not contain any integer other than n itself. If we take some non-integer x, then we can choose $0 < \epsilon < \min\{|x - \lfloor x \rfloor|, |x - \lceil x \rceil|\}$. In this way $B_\epsilon(x)$ does not contain any integer.

Definition 1.2.4 (Closed Sets). Let M be a metric space. We shall call a subset $S \subseteq M$ closed if S contains all its limit points.

Definition 1.2.5 (Perfect Set). A set is called *perfect* if each point of the set is a limit point.

Example 1.2.4

Consider the set

$$S = \left\{ \frac{1}{n} \ : \ n \in \mathbb{N} \right\}$$

This is not a closed set in $\mathbb{R}$. Because 0 is a limit point of S, but $0 \notin S$. If we consider $S \cup \{0\}$, then it's a closed set.

Example 1.2.5

The singleton set $\{1\}$ is a closed set in $\mathbb{R}$. Because it has no limit points, so the set of all limit points is $\varnothing$. And $\varnothing \subseteq \{1\}$.

Example 1.2.6

$\mathbb{N}$ and $\mathbb{Z}$ are also closed in $\mathbb{R}$. Because they have no limit points.

Example 1.2.7

$\mathbb{Q}$ is not closed in $\mathbb{R}$. Because any real number is a limit point of $\mathbb{Q}$, but $\mathbb{Q}$ does not contain any irrational real numbers. In a similar manner, one can show that $\mathbb{R} \setminus \mathbb{Q}$ is also not closed in $\mathbb{R}$.

There is an opposite notion of limit point. It's called isolated point.

Definition 1.2.6 (Isolated Point). Let X be a metric space and $E \subseteq X$ be a nonempty subset of X. Then $p \in X$ is a *isolated point* of E if it's not a limit point. That is, there exists $\epsilon > 0$ such that

$$(E \cap B_\epsilon(p)) \setminus \{p\} = \varnothing$$

Definition 1.2.7 (Dense Set). Let M be a metric space and $E \subseteq M$ be a subset. We shall call E is *dense* in M if every point of M is a limit point of E.

Example 1.2.8

We have seen before that any real number is a limit point of both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$. Therefore, both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$.

§1.3 Compactness

Definition 1.3.1 (Open Cover). Let (X, d) be a metric space. An *open cover* of a set $E \subseteq X$ is a collection of open sets $\{U_\alpha \;:\; \alpha \in I\}$ such that

$$E \subseteq \bigcup_{\alpha \in I} U_\alpha$$

Definition 1.3.2 (Subcover). A subcover of an open cover is just a subcollection of it that covers the set.

Definition 1.3.3 (Compact Set). A set $M \subseteq X$ is called *compact* if every open cover of M in X has a finite subcover.

Example 1.3.1

The singleton set $\{1\}$ is compact. Because you can cover it with only one open set. By a similar argument, every finite set is compact.

Example 1.3.2

The set $S = \left\{\frac{1}{n} \;:\; n \in \mathbb{N}\right\}$ is not compact. Because one can construct a cover of this set that does not have any finite subcover. For example,

$$U_n = B\left(\frac{1}{n}, r_n\right) \quad \text{where} \quad 0 < r_n < \min\left\{\frac{1}{n-1} - \frac{1}{n}, \frac{1}{n} - \frac{1}{n+1}\right\}$$

Every U_n contains exactly one element from S, so there does not exist any finite subcover. However, if we take $S \cup \{0\}$ then it becomes compact. Because 0 is the limit of the sequence $a_n = \frac{1}{n}$, so any open neighborhood of 0 would contain all but finitely many points. Therefore we can cover the all the points with a finite cover.

Example 1.3.3

$\mathbb{N}$ is not compact in $\mathbb{R}$. If we take $U_n = \left(n - \frac{1}{2}, n + \frac{1}{2}\right)$, each U_n contains exactly one element from $\mathbb{N}$. As a result, no finite collection would cover the whole $\mathbb{N}$.

So you can guess that compactness has something to do with closedness. Also intuitively, an unbounded set should not be compact.

Proposition 1.3.1

Compact subsets of a metric space are closed.

Proof. Assume the contrary. Let (X, d) be a metric space and M be a compact subset of X and M is not closed. Thet means there exists a limit point x of M that is not in M. We consider the collection

$$U_n = X \setminus \overline{B(x, 1/n)} = \left\{y \in X \;:\; d(x, y) > \frac{1}{n}\right\}$$

Let $p \in M$, and $r = d(x, p)$. Then there exists some n such that $r > \frac{1}{n}$. Therefore $p \in U_n$. As a result M is contained in $\bigcup U_n$.
Since x is a limit point, U_n contains at least one point in M. If we consider only a finite collection of open sets from $\{U_n\}$, it would not cover the whole M. Therefore, M does not stay compact. Contradiction. ■

Proposition 1.3.2
Closed subsets of a compact set are compact.

Proof. Let M be a compact set and $K \subseteq M$ be closed. We need to show that K is compact. K is closed, so $M \setminus K$ is open. We choose any open cover $\{U_\alpha\}$ of K. Then we have,

$$M \subseteq (M \setminus K) \cup \left(\bigcup U_\alpha\right)$$

Since M is compact, this open cover of M must have a finite subcover. This finite subcover contains a finite collection of $\{U_\alpha\}$, and this finite collection covers K. Therefore, K is compact. ■

Corollary 1.3.3
Suppose F is compact and K is closed in a metric space. Then $F \cap K$ is compact.

Proof. Trivial. ■

Theorem 1.3.4
Suppose $\{K_\alpha\}$ is a collection of compact sets in a metric space such that every finite intersection of sets from this collection is nonempty. Then $\bigcap K_\alpha$ is nonempty.

Theorem 1.3.5 (Heine-Borel)
A set $S \subseteq \mathbb{R}^n$ is compact if and only if it's closed and bounded.

2 Algebra on Complex Numbers

We defined $\mathbb{C}$ to be the field extension of $\mathbb{R}^2$. We showed in Chapter 0 that there must exist some z such that $z^2 = -1$. We denote that number by i. In that way we can write every complex number as $a + ib$, where a and b are real numbers. Now addition and multiplication can be defined on $\mathbb{C}$.

§2.1 Basic Arithmetic

Suppose $z = a + ib$ is a complex number. Then we say a to be the **Real part** of z and denote it by $\operatorname{Re}(z)$; and we say b to be the **Imaginary part** of z and denote it by $\operatorname{Im}(z)$. Now we can define addition and multiplication on $\mathbb{C}$.

> **Definition 2.1.1.** Let $z = a + ib$ and $w = c + id$ be two complex numbers. Then we define addition and multiplication of these two complex numbers as follows:
>
> $$\begin{aligned} z + w &= (a + c) + i(b + d) \\ zw &= (ac - bd) + i(ad + bc) \end{aligned}$$

We choose $0 + 0i$ as the additive identity, and $1 + 0i$ as the multiplicative identity. Then we can also define subtraction (additive inverse) and division (multiplicative inverse).

$$-z = (-a) + i(-b) \text{ and } z^{-1} = \frac{1}{a + ib} = \frac{a - ib}{(a + ib)(a - ib)} = \frac{a}{a + b^2} - i\frac{b}{a^2 + b^2}$$

To see they are inverses indeed,

$$\begin{aligned} z + (-z) &= (a + ib) + ((-a) + i(-b)) = 0 + 0i \\ zz^{-1} &= (a + ib)\left(\frac{a - ib}{(a + ib)(a - ib)} = \frac{a}{a + b^2} - i\frac{b}{a^2 + b^2}\right) \\ &= \left(\frac{a^2}{a^2 + b^2} - \frac{b(-b)}{a^2 + b^2}\right) + i\left(\frac{a(-b)}{a^2 + b^2} + \frac{ab}{a^2 + b^2}\right) \\ &= 1 + 0i \end{aligned}$$

It's easy to check that these definitions are consistent with the other field axioms.
It's often useful to think about the conjugation of a complex number, which is nothing but negating the imaginary part.

> **Definition 2.1.2** (Complex Conjugate)**.** Let $z = a + ib$ be a complex number. Then it's conjugate — denoted by $\overline{z}$ — is defined as follows:
>
> $$\overline{z} = a - ib$$

> **Proposition 2.1.1** (Properties of Conjugate)

Let z_1 and z_2 be complex numbers. Then

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}\ ,\ \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}\ ,\ \overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}\ ,\ \overline{z_1/z_2} = \overline{z_1}/\overline{z_2}$$

Proof. Trivial. ■

§2.2 Geometric Interpretation of Complex Numbers

Since $\mathbb{C}$ is the extension of $\mathbb{R}^2$, we can actually express each complex number as a point in the 2-dimensional plane. The complex number $a + ib$ will be denoted by the point (a, b). The points on the x-axis are of the form $t + 0i$, so they are essentially real numbers. And similarly, the points on the y-axis have real part 0. That's why the x-axis is called the **Real axis**, and the y-axis is called the **Imaginary axis**. Then this plane is called the Complex Plane.

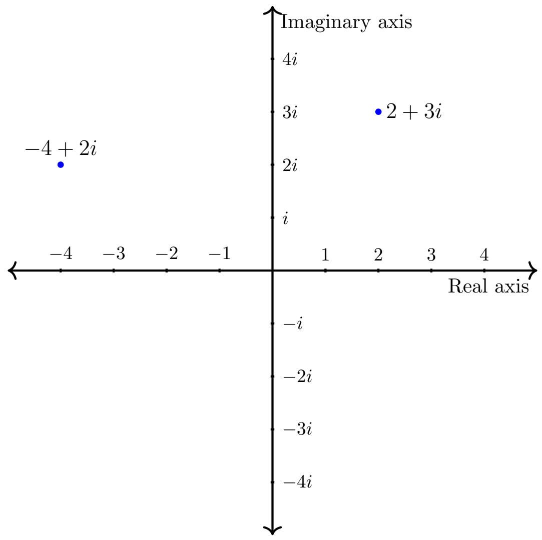


Figure 2.1: The Complex Plane

The distance from the origin to the point is said to be the **modulud** of the complex number.

Definition 2.2.1 (Modulus)**.** Let $z = a + ib$ be a complex number. Then its *modulus* or *magnitude* — denoted by $|z|$ — is defined by

$$|z| := \sqrt{a^2 + b^2}$$

It's easy to check that $z\overline{z} = |z|^2$

Theorem 2.2.1

z_1 and z_2 are two complex numbers, then

$$|z_1| + |z_2| \geq |z_1 + z_2|$$

Proof. One can easily prove it with long and tedious calculation. But there is a visual solution that uses the triangle inequality from Euclidean geometry.
Let $z_1 = a + ib$, $z_2 = c + id$. Let A and B denote the points (a, b) and c, d respectively, and O is the origin. It's clear that $|z_1| = \overline{OA}$ and $|z_2| = \overline{OB}$. Let C be the point

denoting the complex number $z_1 + z_2$. Then C will have coordinates $(a + c, b + d)$. Also, $|z_1 + z_2| = \overline{OC}$. It's easy to check that $OACB$ forms a parallelogram.

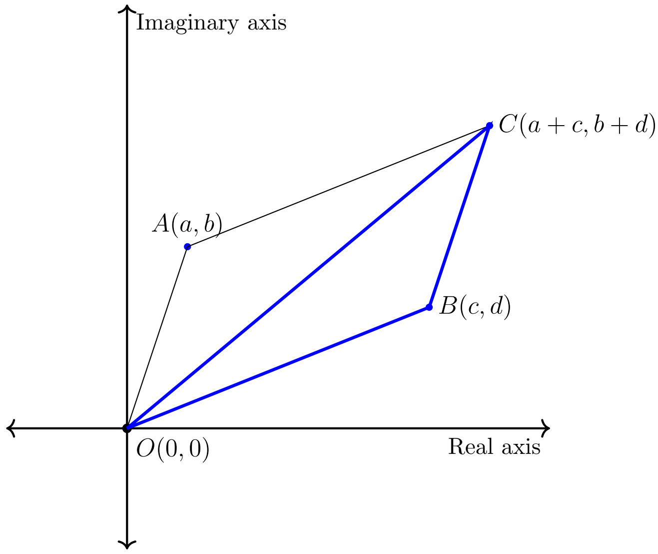


Figure 2.2: Triangle Inequality

Since $OACB$ is a parallelogram, $\overline{OA} = \overline{BC}$. We take the triangle OBC and apply triangle inequality on it.

$$\overline{OB} + \overline{BC} \geq \overline{OC} \implies |z_2| + |z_1| \geq |z_1 + z_2|$$

Hence, we are done. ■

§2.3 Polar Coordinates

Cartesian coordinate and polar coordinate comes hand in hand. We've seen before that the complex number $z = x + iy$ can be expressed on the cartesian plane by the point (x, y). The distance from the origin to this point is called the modulus of z. If we join the origin and the point x, y, this segment creates some angle with the positive real axis. This angle is called *argument* of z.

Definition 2.3.1 (Argument)**.** Let $z = x + iy$ be a complex number, and $|z| = r$. Then it's argument $\arg z$ is a real number θ for which $x = r\cos\theta$ and $y = r\sin\theta$ are satisfied. The *principal argument* — denoted by $\operatorname{Arg} z$ — is an argument that lies in $[0, 2\pi)$.

Let $z = x + iy$, $r = |z|$, $\theta = \arg(z)$. So

$$z = r\cos\theta + i\ r\sin\theta = r\left(\cos\theta + i\sin\theta\right)$$

This gives us a geometric interpretation for complex multiplication.

Proposition 2.3.1

$|z_1 z_2| = |z_1|\,|z_2|$ and $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$, where the sum is taken modulo 2π.

Proof. Let $r_1 = |z_1|$, $\theta_1 = \text{Arg}\,(z_1)$ and $r_2 = |z_2|$, $\theta_2 = \text{Arg}\,(z_2)$.

$$\begin{aligned} z_1 z_2 &= r_1 (\cos\theta_1 + i\sin\theta_1)\, r_2 (\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2) \\ &= r_1 r_2 \big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big) \end{aligned}$$

Therefore, $|z_1 z_2| = r_1 r_2 = |z_1|\,|z_2|$, and $\text{Arg}\,(z_1 z_2) = (\theta_1 + \theta_2) = \text{Arg}\,(z_1) + \text{Arg}\,(z_2)$. ■

This means, when you multiply one complex number by another, you're basically scaling the length and then rotating it.

Now, what's the attribute of a number that gets added when you multiply? The simplest such example one can think of is exponent. That is, $e^a \cdot e^b = e^{a+b}$. Euler showed that, argument of complex number really works like exponents.

Proposition 2.3.2 (Euler's formula)
$e^{i\theta} = \cos\theta + i\sin\theta$

Proof. Complex exponent is defined by the Mclaurin series for e^x.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Substituting $x = i\theta$,

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(i\theta)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(i\theta)^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(\theta)^{2k}}{(2k)!} + i\sum_{k=0}^{\infty} (-1)^k \frac{(\theta)^{2k+1}}{(2k+1)!} \\ &= \cos\theta + i\sin\theta \end{aligned}$$

■

Corollary 2.3.3 (De Moivre's Theorem)
$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

De Moivre proved it way before Euler's formula, and that proof is a bit complicated. But this is just a simple exercise using Euler's formula.

§2.3.i Branch of Argument

On definition 2.3.1 we defined the principal argument $\text{Arg}(z)$ to be the unique argument that lies in $[0, 2\pi)$. However, this is not the only way to define the arg function. We know that, if θ is an argument of a complex number z, then so is $\theta + 2n\pi$ for $n \in \mathbb{Z}$. Depending on our need, we can take any interval of length 2π, with one side closed and the other side open, and define argument function that gives output on that "half-open-half-closed" interval. This is called branch of argument.

For example, suppose we are defining argument on the branch $(-\pi, \pi]$. In that case, $\arg(-i) = -\frac{\pi}{2}$, and $\arg\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$. Or we could have defined it on the branch $\left[\frac{\pi}{4}, \frac{9\pi}{4}\right)$.

This idea is extremely useful when we work with complex logarithm. We've seen that, if z is a complex number,

$$z = |z|\, e^{i\arg(z)}$$

If we take log on both sides,

$$\log(z) = \log\left(|z|\right) + \log\left(e^{i\arg(z)}\right) \log\left(|z|\right) + i\arg(z)$$

Depending on what range of logarithm we are seeking, we choose branch of argument accordingly.

§2.4 Exponentiation and Roots

Euler's formula gives us a way to define exponents and roots in $\mathbb{C}$. Suppose $z = x + iy$ is a complex number. Then we have

$$e^z = e^{x+iy} = e^x\ e^{iy} = e^x\left(\cos y + i\sin y\right)$$

Also, the modulus of e^z is e^x, and the argument of e^z is y.

Now we want to find the n-th roots of a complex number c. Algebraically this means solving the equation

$$z^n = c$$

Let $c = Re^{i\phi}$ and $z = re^{i\theta}$. We need to find r and θ.

$$\begin{aligned} z^n &= \left(re^{i\theta}\right)^n = r^n e^{i(n\theta)} \\ \implies Re^{i\phi} &= r^n e^{i(n\theta)} \\ \implies r &= R^{1/n} \text{ and } n\theta = \phi + 2k\pi\ , \quad \text{where } k \in \mathbb{Z} \\ \implies r &= R^{1/n} \text{ and } \theta \in \left\{\frac{\phi}{n} + \frac{2k\pi}{n}\ :\ k \in \mathbb{Z}\right\} \end{aligned}$$

In this way one can easily compute the n-th roots of a complex number c. However, when $c = 1$ the roots are called **roots of unity**. When $c = 1$, we have $R = 1$ and $\phi = 0$. Therefore, $r = 1$, and

$$\theta \in \left\{\frac{2k\pi}{n}\ :\ 0 \le k \le n - 1\right\}$$

So all the n-th roots of unity are of the form $e^{i\frac{2k\pi}{n}}$. So it is evident that the roots will lie evenly on the unit circle. Figure 2.3 contains an example with $n = 5$.

§2.5 Matrix Representation of Complex Numbers

Let R_θ be the rotation of the plane $\mathbb{R}^2$ that rotates with an angle of θ. By definition, it is a linear map. So, how can we write the map explicitly?

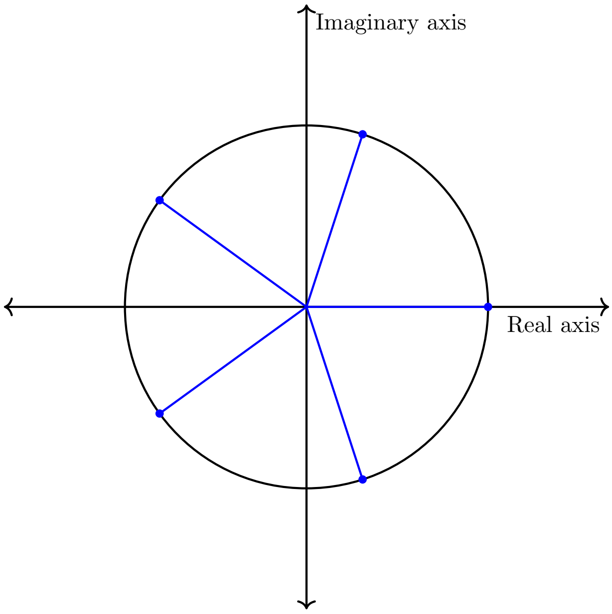


Figure 2.3: 5-th roots of unity on unit circle

First of all, the question doesn't make any sense if we don't have any basis. So we need a basis for $\mathbb{R}^2$. Let's just take the usual basis $\mathcal{B} = \{e_1 = (1,0), e_2 = (0,1)\}$. Then

$$\begin{aligned} R_\theta((x,y)) &= R_\theta(x(1,0) + y(0,1)) = xR_\theta((1,0)) + yR_\theta((0,1)) \\ &= x(\cos\theta, \sin\theta) + y(-\sin\theta, \cos\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) \end{aligned}$$

Observe that, each such rotation R_θ can be presented by multiplication of a complex number $z_\theta \in \mathbb{C}$, where $z_\theta = \cos\theta + i\sin\theta$. Because

$$z_\theta(x+iy) = (\cos\theta + i\sin\theta)(x+iy) = (x\cos\theta - y\sin\theta) + i(x\sin\theta + y\cos\theta)$$

Note that, $R_\theta \circ R_\phi = R_\phi \circ R_\theta$. Each R_θ represents multiplication by a 2×2 matrix. Usually matrix multiplication is not commutative. But why does it happen here?

The answer lies in complex plane. R_θ is associated with multiplication by a complex number z_θ. Since $\mathbb{C}$ is a field, multiplication is commutative. That's why $R_\theta \circ R_\phi = R_\phi \circ R_\theta$ happens.

The set of all such R_θ forms a group, it is denoted by $SO(2)$. The set $\{z \in \mathbb{C} : |z| = 1\}$ forms a group denoted by S^1. Then S^1 is isomorphic to $SO(2)$. An explicit isomorphism is

$$\cos\theta + i\sin\theta \mapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Abusing notation, let R_θ be the same 2×2 matrix that represents the linear map.

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \cos\theta\ \mathbf{1} + \sin\theta\ \mathbf{i}$$

These $\mathbf{1}$ and $\mathbf{i}$ behave like $1, i \in \mathbb{C}$ (because $\mathbf{i}^2 = -\mathbf{1}$, and $\mathbf{1}$ is the multiplicative identity). This is another way of convincing yourself that z_θ and R_θ are basically the same.

If we forget about the condition $|z| = 1$,

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = a\ \mathbf{1} + b\ \mathbf{i}$$

So, there is a one-to-one correspondence between $\mathbb{C}$ and the set of all 2×2 real matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

In this correspondence, the square of the norm corresponds to determinant of the matrix, because

$$\begin{vmatrix} a & -b \\ b & a \end{vmatrix} = a^2 + b^2 = |a + ib|^2$$

And the multiplicative inverse of a complex number corresponds to the inverse matrix, since

$$(a + ib)^{-1} = \frac{a - ib}{a^2 + b^2}, \quad \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} = \frac{1}{a^2 + b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

§2.6 Quaternions

In the previous section, we represented complex numbers with 2×2 real matrices.

$$a + ib \longleftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

What if we allow the entries to be complex numbers? If we take the set $\mathbb{H}$ of all 2×2 complex matrices of the form

$$\begin{bmatrix} z_1 & -z_2 \\ \overline{z_2} & \overline{z_1} \end{bmatrix},$$

then $\mathbb{H}$ forms an abelian group under matrix addition. Also $\mathbb{H} \setminus \{\mathbf{0}\}$ forms a group under matrix multiplication (multiplication is, in general, not commutative). So $\mathbb{H}$ is not a field. This $\mathbb{H}$ is called the set of all **quaternions**.

If $q \in \mathbb{H}$,

$$q = \begin{bmatrix} a + id & -b - ic \\ b - ic & a - id \end{bmatrix} = a\,\mathbf{1} + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$$

$$\text{where } \mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathbf{i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ \mathbf{j} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \ \mathbf{k} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

Verify that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}$ and $\mathbf{ij} = \mathbf{k}, \mathbf{jk} = \mathbf{i}, \mathbf{ki} = \mathbf{j}$. Also, we can define the norm as follows:

$$\det q = a^2 + b^2 + c^2 + d^2 =: |q|^2$$

Furthermore, we can define the conjugate of q:

$$q = a\,\mathbf{1} + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k} \implies \overline{q} = a\,\mathbf{1} - b\,\mathbf{i} - c\,\mathbf{j} - d\,\mathbf{k}$$

Using the corresponding matrix inverse, one can verify that

$$q^{-1} = \frac{1}{a^2 + b^2 + c^2 + d^2} (a\,\mathbf{1} - b\,\mathbf{i} - c\,\mathbf{j} - d\,\mathbf{k})$$

This gives us the identity $q\overline{q} = \overline{q}q = |q|^2$. But the conjugate of product is different from what we can expect.

$$\overline{q_1 q_2} = \overline{q_2}\,\overline{q_1}$$

Likewise unit complex numbers, the set of all unit quaternions form a group uner multiplication. It is denoted by S^3.

$$S^3 = \{q \in \mathbb{H} \,:\, |q| = 1\}$$

We will see later that this S^3 helps us understand rotations in $\mathbb{R}^3$.

§2.7 Rotations

Let $v, w \in \mathbb{C}$, then for any $u \in S^1$ we have

$$|v - w| = |vu - wu|$$

This means, multiplication by u is an isometry (distance preserving map). Also, note that this isometry fixes the origin. There is a unique such isometry in $\mathbb{R}^2$, it's a rotation.

Similarly, for any $u, w \in \mathbb{R}^4$,

$$|u - w| = |uq - wq| \quad \text{for } q \in S^3.$$

We can explain rotation in $\mathbb{R}^3$ similarly, but we need to be a bit tricky.

If a quaternion has 0 as the coefficient of $\mathbf{1}$, we shall call it a **"pure imaginary"** quaternion. We can consider $\mathbb{R}^3$ to be the set of all pure imaginary quaternions.

$$\mathbb{R}^3 = \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{j}$$

If $u = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{j}$ and $v = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{j}$ are two pure imaginary quaternions, the product of them has a nice representation. Let $u \cdot v$ be the dot product of the vectors u and v. Likewise, $u \times v$ is the cross product. The quaternion product is uv. Then

$$uv = -(u \cdot v)\,\mathbf{1} + (u \times v)$$

This shows that, $uv \in \mathbb{R}^3$ iff $u \cdot v = 0$, which is equivalent to u and v being perpendicular. For unit u, $u \times u = 0$ gives us $u^2 = -(u \cdot u) = -1$. Thus we get, $x^2 + 1 = 0$ has uncountably many solutions in quaternions. This is a counterexample of fundamental theorem of algebra (FTA). In general, since $\mathbb{H}$ is not a field, FTA does not hold in $\mathbb{H}$.

Theorem 2.7.1

Let $t = \cos\theta + u\sin\theta$, where u is a pure imaginary quaternion ($u \in \mathbb{R}^3$). Consider the map $f : \mathbb{H} \to \mathbb{H}$ given by

$$q \mapsto tqt^{-1}.$$

Then f rotates $\mathbb{R}^3 = \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{j}$ through angle 2θ about u.

Sketch of proof. If $r \in \mathbb{R}$, then

$$f(r) = trt^{-1} = r$$

So f preserves the real line. Now choose a unit vector v with $v \perp u$. Then

$$w = u \times v = uv.$$

So $\{u, v, w\}$ forms a basis. Then show that

$$tvt^{-1} = v\cos 2\theta - w\sin 2\theta \quad \text{and} \quad twt^{-1} = v\sin 2\theta + w\cos 2\theta$$

Also, f preserves all point on the line through u. So using these, one can easily conclude the result. ∎

This is a ground-breaking result in game development and computer graphics. Previously game engines used to use Euler angles to portray rotation. But that leads to some inconveniences, such as Gimbal lock. Rather, this method just requires multiplication by two quaternions. This is not only computationally efficient, but also free from all sorts of ambiguity. More about that in th following 3blue1brown video: https://youtu.be/zjMuIxRvygQ

3 Differentiation

As a vector space $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$. One can think of the ordered pair (x, y) as a way of representing the complex number $x + iy$. But the notion of linear map is not the same in $\mathbb{C}$ and $\mathbb{R}^2$. In $\mathbb{C}$, the underlying field for the vector space is $\mathbb{C}$. On the other hand, the underlying field for the vector space $\mathbb{R}^2$ is $\mathbb{R}$.

Recall the definition of linear maps. A linear map preserves the notion of vector addition and scalar multiplication between vector spaces. In other words, a map T between vector spaces is called linear if for every vector v_1, v_2 and scalar c,

$$T(v_1 + v_2) = T(v_1) + T(v_2) \text{ and } T(cv_1) = cT(v_1)$$

Using this, we can define linear map in $\mathbb{C}$.

Definition 3.0.1 (Linear Map). A map $T : \mathbb{C} \to \mathbb{C}$ is called **$\mathbb{R}$-linear** if for every $z, w \in \mathbb{C}$ and $c \in \mathbb{R}$

$$T(z + w) = T(z) + T(w) \text{ and } T(cz) = cT(z)$$

We call the map **$\mathbb{C}$-linear** if this equation is satisfies for every $c \in \mathbb{C}$ too.

Proposition 3.0.1

Let $T : \mathbb{C} \to \mathbb{C}$ be $\mathbb{R}$-linear. Then it's $\mathbb{C}$-linear if and only if $T(iz) = iT(z)$ for every $z \in \mathbb{C}$.

Proof. ($\Rightarrow$) Assume T is $\mathbb{C}$-linear. Then we choose $c = i$ and $T(iz) = iT(z)$ follows immediately.

($\Leftarrow$) Now assume $T(iz) = iT(z)$ for every $z \in \mathbb{C}$. We need to show that $T(cz) = cT(z)$ for every $c, z \in \mathbb{C}$. Let $c = a + ib$ where $a, b \in \mathbb{R}$.

$$\begin{aligned} T(cz) &= T((a + ib)z) = T(az + ibz) \\ &= T(az) + T(ibz) = aT(z) + bT(iz) \\ &= aT(z) + ibT(z) = (a + ib)T(z) = cT(z) \end{aligned}$$

So T is a $\mathbb{C}$-linear map. ∎

In general, for any real vector space V, if we want to make it complex vector space, the only thing that determines the $\mathbb{C}$-vector space is $i\mathbf{e_j}$; where $\mathbf{e_j}$ is a basis. In other words, how the scalar i acts on the vector space.

Lemma 3.0.2

If $T : \mathbb{C} \to \mathbb{C}$ is a $\mathbb{C}$-linear map, then $T(z) = cz$ for some fixed $c \in \mathbb{C}$.

Proof. Let $c = T(1)$. $T(z) = T(z \cdot 1) = z \cdot T(1) = cz$. ∎

§3.1 Differentiability

Complex differentiability is very much similar to differentiability in $\mathbb{R}$.

> **Definition 3.1.1** (Differentiable Function)**.** Suppose V is an open subset of $\mathbb{C}$ and $z \in \mathbb{C}$. We call a function $f : V \to \mathbb{C}$ **differentiable** at z_0 if the following limit exists:
> $$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

The limit can also be written in the following way:

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

We often denote this differentiability by $\mathbb{C}$-differentiable. Here we need to address the significance of V being open. Since V is open, given $z_0 \in V$ we can find a open ball $B(z_0, r) \subseteq V$. That is $z_0 + h \in V$ whenever $|h| < r$. So while computing the limit, $z_0 + h$ is free to approach z_0 from **any direction** as $h \to 0$. If we get two different values of the limit while approaching $h \to 0$ via two different directions, then the limit does not exist.

Since $\mathbb{C} \cong \mathbb{R}^2$, we can compare $\mathbb{C}$-differentiability with $\mathbb{R}^2$-differentiability. Recall that, if V is an open subset of $\mathbb{R}^2$, then a function $f : V \to \mathbb{R}^2$ is said to be *differentiable* at $z \in V$ if there exists an $\mathbb{R}$ linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f(z + h) = f(z) + Th + o(h) \quad \text{as } h \to 0$$

In this case, the operator T is known as the *Frechet derivative* of f at z. It's also written as $Df(z)$. Here $o(h)$ basically means the error term, that satisfies $o(0) = 0$ and

$$\lim_{h \to 0} \frac{\|o(h)\|}{\|h\|} = 0$$

For now we shall call such a function $\mathbb{R}$-differentiable. Using the limiting property of $o(h)$, we can rewrite $\mathbb{R}$-differentiability as follows:

$$\frac{\|f(z + h) - f(z) - Th\|}{\|h\|} \to 0 \quad \text{as } h \to 0$$

> **Proposition 3.1.1**
> Suppose V is a open subset of $\mathbb{C}$, $f : V \to \mathbb{C}$ and $z \in V$. Then f is $\mathbb{C}$-differentiable at $z \in V$ if and only if f is $\mathbb{R}$-differentiable at z and the map $Df(z)$ is complex linear.

Proof. ($\Rightarrow$) Assume f is $\mathbb{C}$-differentiable at z.

$$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} \implies f(z + h) = f(z) + f'(z)h + o(h) \quad \text{as } h \to 0$$

The map $Df(z)(h) = f'(z)h$ is clearly $\mathbb{C}$-linear, so it is also $\mathbb{R}$-linear. As a result f is $\mathbb{R}$-differentiable.

($\Leftarrow$) Now assume f is $\mathbb{R}$-differentiable at z, with $Df(z)$ being a $\mathbb{C}$-linear map. By Lemma 3.0.2 , $Df(z)(h) = ch$ for some $c \in \mathbb{C}$. Therefore, by the $\mathbb{R}$-differentiability condition,

$$\begin{aligned} & f(z+h) = f(z) + Df(z)\,(h) + o(h) \quad \text{as } h \to 0 \\ \implies & f(z+h) = f(z) + ch + o(h) \quad \text{as } h \to 0 \\ \implies & c = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \end{aligned}$$

So the limit exists and f is $\mathbb{C}$-differentiable at z. ∎

§3.1.i Cauchy Riemann Equations

Let $f : V \to \mathbb{C}$, where $V \subseteq \mathbb{C}$ is open. We write $f = u + iv$, where $u = \operatorname{Re}(f) : \mathbb{R}^2 \to \mathbb{R}$ and $v = \operatorname{Im}(f) : \mathbb{R}^2 \to \mathbb{R}$. We define the partial derivatives of u and v in the usual way:

$$u_x = \frac{\partial u}{\partial x}\ ,\ u_y = \frac{\partial u}{\partial y} \quad \text{and} \quad v_x = \frac{\partial v}{\partial x}\ ,\ v_y = \frac{\partial v}{\partial y}$$

Then the **Cauchy-Riemann equations**, or **CR equations** in short, are

$$\boxed{u_x = v_y \quad \text{and} \quad u_y = -v_x} \tag{3.1}$$

Proposition 3.1.2

Let $V \subseteq \mathbb{C}$ be open. $f : V \to \mathbb{C}$ and $z \in V$. Then f is $\mathbb{C}$-differentiable at z if and only if f is $\mathbb{R}$-differentiable at z and the CR equations hold at z.

Proof. f is $\mathbb{R}$-differentiable at z, so $Df(z)$ represents a $\mathbb{R}$-linear map described by the 2×2 matrix

$$Df(z) \sim \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

By Proposition 3.1.1, we only need to verify that $Df(z)$ being a $\mathbb{C}$-linear map is equivalent to satisfying CR equations at z. Also by Proposition 3.0.1, $Df(z)$ being a $\mathbb{C}$-linear map is equivalent to $Df(z)\,(ic) = i\ Df(z)\,(c)$. So it's enough for us to show that $Df(z)\,(ic) = i\ Df(z)\,(c)$ is equivalent to satisfying CR equations. Let $c = x_1 + iy_1$. We've seen earlier that, multiplication by i has the same effect as multiplicatio by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Therefore,

$$\begin{aligned} Df(z)\,(ic) = i\ Df(z)\,(c) \iff & \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \\ \iff & \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \\ \iff & \begin{bmatrix} u_y & -u_x \\ v_y & -v_x \end{bmatrix} = \begin{bmatrix} -v_x & -v_y \\ u_x & u_y \end{bmatrix} \\ \iff & u_x = v_y \quad \text{and} \quad u_y = -v_x \end{aligned}$$

So we are done! ∎

Corollary 3.1.3
Let $V \subseteq \mathbb{C}$ be open. $f : V \to \mathbb{C}$ and $z \in V$. If f is $\mathbb{C}$-differentiable at z, then CR equations are satisfied at z.

This corollary shows us that CR equations are a necessary condition for differentiability. However, it is **NOT** a sufficient condition.

Example 3.1.1
Consider the function

$$f(x+iy) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0 \\ 1 & \text{otherwise} \end{cases}$$

In this case, v is always 0; $u = 0$ if $x = 0$ or $y = 0$ and $u = 1$ otherwise. So we have

$$u_x = u_y = v_x = v_y = 0$$

But f is not differentiable at 0. Because the limit $\lim_{h\to 0} \frac{f(h)-f(0)}{h}$ fails to exist. To prove this, one can just approach $h \to 0$ via x-axis and the identity line $x = y$. The former one gives the limit 0, while the latter one yields 1.

Corollary 3.1.4
Let $V \subseteq \mathbb{C}$ be open. $f : V \to \mathbb{C}$ and $z \in V$. Then f is $\mathbb{C}$-differentiable at z if and only if the partial derivatives u_x, u_y, v_x, v_y are continuous at z and the CR equations hold at z.

Proof. From *Calculus of several variables*, we know that having continuous first order partial derivatives is equivalent to being $\mathbb{R}$-differentiable. Then by Proposition 3.1.2, we are done! ∎

Proposition 3.1.5
If $f : V \to \mathbb{C}$ is $\mathbb{C}$-differentiable at $z_0 \in V$, then f is continuous at z_0.

Proof. We wish to prove that $\lim\limits_{z\to z_0} |f(z) - f(z_0)| = 0$.

$$\lim_{z\to z_0} |f(z) - f(z_0)| = \left(\lim_{z\to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}\right)\left(\lim_{z\to z_0} |z - z_0|\right) = |f'(z_0)| \cdot 0 = 0$$

So f is continuous at z_0. ∎

Definition 3.1.2 (Holomorphic Function)**.** We call a function $f : \mathbb{C} \to \mathbb{C}$ **holomorphic** at $z_0 \in \mathbb{C}$ if it is differentiable at all points in a neighborhood of z_0. We call f holomorphic on a set V if it is differentiable at all points of V.

Holomorphicity is clearly a stronger condition than differentiability. There are functions that are differentiable at a point, but not differentiable at any neighborhood of that point.

Example 3.1.2

Consider the function $f(z) = |z|^2 = x^2 + y^2$. f is differentiable on $z = 0$, but not at any neighborhood of 0 (check this using CR equations).

Proposition 3.1.6 (Chain Rule)

Let f and g be holomorphic on G and Ω respectively and suppose $f(G) \subseteq \Omega$. Then $g \circ f$ is holomorphic on G and

$$(g \circ f)'(z) = g'(f(z))\, f'(z)$$

for all $z \in G$.

Proof. Since f and g are differentiable on G and Ω respectively,

$$f(z+h) = f(z) + f'(z)h + o(h) \quad \text{and} \quad g(w+h) = g(w) + g'(w)h + o(h) \quad \text{for } z \in G,\ w \in \Omega$$

Let $z_0 \in G$. Wre wish to compute $(g \circ f)'(z_0)$.

$$\begin{aligned}
(g \circ f)'(z_0) &= \lim_{h \to 0} \frac{g(f(z_0+h)) - g(f(z_0))}{h} \\
&= \lim_{h \to 0} \frac{g(f(z_0) + f'(z_0)h + o(h)) - g(f(z_0))}{h} \\
&= \lim_{h \to 0} \frac{g(f(z_0)) + g'(f(z_0))(f'(z_0)h + o(h)) + o(f'(z_0)h + o(h)) - g(f(z_0))}{h} \\
&= g'(f(z_0)) \lim_{h \to 0} \frac{f'(z_0)h + o(h)}{h} + \lim_{h \to 0} \frac{o(f'(z_0)h + o(h))}{h} \\
&= g'(f(z_0))\, f'(z_0)
\end{aligned}$$

as desired. ■

§3.2 Wirtinger Derivative Operators

Definition 3.2.1 (Wirtinger Derivative). Let $f : V \to \mathbb{C}$ be a fucntion. The Wirtinger derivatives are **defined** as the following linear partial differential operators:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Previously we considered f to be a function of x and y and we partially differentiated with respect to x and y to get Cauchy-Riemann equations. Now we are basically considering f to be a function of z and $\bar{z}$. There is a catch though. Unline x and y, z and $\bar{z}$ are not really independent. So the partial derivative doesn't make any sense. But this newly defined operators kinda "behave like" partial derivatives.

This operators are very useful. If f is differentiable, then $\frac{\partial f}{\partial z}$ is exactly equal to $f'(z)$. Also, Cauchy-Riemann equations can be rewritten using this notation $\frac{\partial f}{\partial \bar{z}} = 0$.

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial(u+iv)}{\partial x} - i\frac{\partial(u+iv)}{\partial y}\right)\\ &= \frac{1}{2}\left(u_x + iv_x - iu_y - i^2 v_y\right) = \frac{1}{2}\left(u_x + iv_x - iu_y + v_y\right)\\ &= u_x + iv_x = f'(z)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial(u+iv)}{\partial x} + i\frac{\partial(u+iv)}{\partial y}\right)\\ &= \frac{1}{2}\left(u_x + iv_x + iu_y + i^2 v_y\right) = \frac{1}{2}\left((u_x - v_y) + i\left(u_y + v_x\right)\right)\\ &= 0\end{aligned}$$

Sometimes it's much more convenient to check $\frac{\partial f}{\partial \bar{z}} = 0$ than checking Cauchy-Riemann equations. Even though z and $\overline{z}$ are not independent variables, the Wirtinger derivatives treat them like independent variables. It's easy to check that

$$\frac{\partial z}{\partial z} = 1 \text{ and } \frac{\partial \overline{z}}{\partial z} = 0\ , \qquad \frac{\partial \overline{z}}{\partial \overline{z}} = 1 \text{ and } \frac{\partial z}{\partial \overline{z}} = 0$$

That's why Wirtinger derivatives behave like partial derivatives.

4 Power Series

What do we mean when we say $\sum_{n=1}^{\infty} x_n$ converges, where $x_n \in X$ for some space X? Note that this question makes absolutely no sense when X is an arbitrary space, say random topological space. Because the operation of addition might not be defined in an arbitrary space. This is not the only reason; we can't talk about convergence untill we have a topological space.

But the good news is, our space $\mathbb{C}$ is a super nice guy. Meaning $\mathbb{C}$ is a 2-dimensional real vector space, so we have a notion of addition here. Moreover, there is a notion of norm (modulus); with help of this norm we can make $\mathbb{C}$ a metric space. So we can just relax and talk about convergence here.

Definition 4.0.1 (Convergence and Absolute convergence)**.** If $a_n \in \mathbb{C}$ for every $n \geq 0$, then the series $\sum a_n$ **converges** to z iff for every $\varepsilon > 0$, there is an integer N such that

$$\left|\sum_{n=0}^{m} a_n - z\right| < \varepsilon \quad \text{whenever } m \geq N$$

The series $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges.

Lemma 4.0.1

$\mathbb{C}$ is complete, *i.e*, every Cauchy sequence in $\mathbb{C}$ has a limit in $\mathbb{C}$.

Proof. Let $z_n = x_n + iy_n$ be a Cauchy sequence, with $x_n, y_n \in \mathbb{R}$. We claim that both x_n and y_n are Cauchy sequences in $\mathbb{R}$. Let $\varepsilon > 0$. Since z_n is Cauchy, we can find $k, m \in \mathbb{N}$ such that $|z_m - z_k| < \varepsilon$. Now,

$$\varepsilon^2 > |z_m - z_k|^2 = |x_m - x_k|^2 + |y_m - y_k|^2 \implies |x_m - x_k|^2 \leq \varepsilon^2 \implies |x_m - x_k| \leq \varepsilon$$

Hence x_n is a Cauchy sequence in $\mathbb{R}$. Similarly y_n is also a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $x_n \to x$ and $y_n \to y$. Therefore,

$$z_n = x_n + iy_n \to x + iy \in \mathbb{C}$$

So $\mathbb{C}$ is complete. ■

Proposition 4.0.2

If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Proof. Let $\varepsilon > 0$ and $z_n = a_0 + a_1 + \cdots + a_n$. Since $\sum |a_n|$ converges, there exists a positive integer N such that $\sum_{n=N}^{\infty} |a_n| < \varepsilon$. Now, for $m > k \geq N$,

$$|z_m - z_k| = \left|\sum_{n=k+1}^{m} a_n\right| \leq \sum_{n=k+1}^{m} |a_n| \leq \sum_{n=N}^{\infty} |a_n| < \varepsilon$$

So $(z_n)_{n=0}^{\infty}$ is a Cauchy sequence. Since $\mathbb{C}$ is complete, this sequence has a limit in $\mathbb{C}$. Hence $\sum a_n$ converges. ■

§4.1 Power Series

Definition 4.1.1 (Power Series). A power series is a series of the form $\sum_{n=0}^{\infty} a_n (z-a)^n$, where $a_n, a \in \mathbb{C}$

Let's talk about its convergence. This power series has a beautiful property. For the sake of simplicity, let's assume $a = 0$. Then our power series is $\sum_{n=0}^{\infty} a_n z^n$.

Proposition 4.1.1

Suppose there exists $w \in \mathbb{C}$ such that $\sum_{n=0}^{\infty} a_n w^n$ converges. Then for any $z \in \mathbb{C}$ with $|z| < |w|$, $\sum_{n=0}^{\infty} a_n z^n$ converges.

Proof. We claim that $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely. Notice that the sequence $x_k = \sum_{n=0}^{k} |a_n z^n|$ is monotonically increasing. So it converges if and only if it is bounded. Hence we only need to prove that $\sum_{n=0}^{\infty} |a_n z^n|$ is bounded.

Since $\sum_{n=0}^{\infty} a_n w^n$ converges, there exists a real number M such that $|a_n w^n| \leq M$ for every n. Now,

$$\sum_{n=0}^{\infty} |a_n z^n| = \sum_{n=0}^{\infty} |a_n w^n| \left|\frac{z}{w}\right|^n \leq \sum_{n=0}^{\infty} M \left(\frac{|z|}{|w|}\right)^n = M \sum_{n=0}^{\infty} \left(\frac{|z|}{|w|}\right)^n$$

By the given condition, $\frac{|z|}{|w|} < 1$. So $\sum_{n=0}^{\infty} \left(\frac{|z|}{|w|}\right)^n$ is a convergent geometric series. Hence $\sum_{n=0}^{\infty} |a_n z^n|$ is bounded, and we are done. ∎

Observe that, $\sum_{n=0}^{\infty} a_n w^n$ convergent is a much stronger condition than $|a_n w^n| \leq M$. Because, convergence of $\sum_{n=0}^{\infty} a_n w^n$ automatically implies $|a_n w^n| \leq M$; but the converse is not always true. For example, $\left|\frac{1}{n}\right| \leq 1$ for all n, but $\sum \frac{1}{n}$ diverges. However, this weaker condition of $|a_n w^n| \leq M$ is enough for us to prove Proposition 4.1.1.

Now, what's the moral of Proposition 4.1.1? It gives us an intuitive idea about the concept of "radius of convergence". If there exists a point on which the power series converges, then the series converges on a disk. Later we will see that *radius of convergence* is the radius of the maximal such disk. Also this result tell us about the "bounded implies convergent" property of power series. It motivates the following definition.

Definition 4.1.2 (Radius of Convergence). Suppose $(c_n)_{n=0}^{\infty}$ is a sequence of complex numbers. We define $R \in [0, \infty]$ by

$$R := \sup \{ r \geq 0 \ : \ \text{the sequence } (c_n r^n)_{n=0}^{\infty} \text{ is bounded} \}$$

We call R the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n (z-z_0)^n$.

Proposition 4.1.2

$\sum_{n=0}^{\infty} c_n (z-z_0)^n$ converges absolutely and uniformly on every compact subset of the disk $D(z_0, R)$; and diverges at every point z with $|z - z_0| > R$.

Proof. The second part follows immediately. If $|z - z_0| > R$, then the terms $c_n (z-z_0)^n$ aren't even bounded. So the summation $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ clearly diverges.

But the first part is a bit trickier. Because the compact subsets of $D(z_0, R)$ might be (are likely to be) weird and ugly sets. And working with those ugly sets is gonna be difficult. Instead we will prove a stronger result. Notice that, if S is a compact subset of $D(z_0, R)$, then S must be contained in some closed ball $\bar{D}(z_0, r)$ for some $r < R$. So we now want to show that the power series converges uniformly and absolutely in $\bar{D}(z_0, r)$.

As $r < R$, we can choose a number ρ such that $r < \rho < R$ (if you're not sure about how to choose that number, or you need an explicit construction of ρ, just take $\rho = \frac{r+R}{2}$). By the definition of R, since $\rho < R$, $c_n\rho^n$ is bounded. That is, $|c_n|\,\rho^n \leq A$ for every n. Now for any $z \in \bar{D}(z_0, r)$,

$$\sum_{n=0}^{\infty} c_n (z-z_0)^n \leq \sum_{n=0}^{\infty} c_n r^n \leq= \sum_{n=0}^{\infty} c_n \rho^n \left(\frac{r}{\rho}\right)^n \leq \sum_{n=0}^{\infty} A \left(\frac{r}{\rho}\right)^n = A \sum_{n=0}^{\infty} \left(\frac{r}{\rho}\right)^n$$

As $r < \rho$, $\sum_{n=0}^{\infty} \left(\frac{r}{\rho}\right)^n$ is a convergent geometric series, hence it is pointwise absolutely convergent in $\bar{D}(z_0, r)$. Observe that, $\bar{D}(z_0, r)$ itself is a compact set. By Dini's theorem, pointwise absolute convergence of monotone sequence of functions on a compact set is uniform convergence. Hence, we are done! ■

So, the radius of convergence is indeed the radius of the maximal disk on which the power series converges. In other words,

$$R = \sup \left\{ |z - z_0| \ : \ \sum_{n=0}^{\infty} c_n (z-z_0)^n \ \text{converges} \right\}$$

But this definition is not good for finding an explicit value of R. For that, we need the following lemma.

Lemma 4.1.3

Let $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ be a power series and R be its radius of convergence. Then the following are true:

(a) $\frac{1}{R} = \limsup |c_n|^{\frac{1}{n}}$.

(b) $\frac{1}{R} = \liminf |c_n|^{\frac{1}{n}}$.

(c) $R = \lim\limits_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$ if the limit exists.

Proof. (a) Let $\frac{1}{R'} = \limsup |c_n|^{\frac{1}{n}}$. We can assume $R' < \infty$, because if $R' = \infty$, then it can easily be shown that R must also be ∞. WLOG, we assume $z_0 = 0$. So we wish to prove that $\sum_{n=0}^{\infty} c_n z^n$ converges if $|z| < R'$, and diverges if $|z| > R'$. That would prove that R' is indeed the radius of convergence.

Assume $|z| < R'$. Let $\varepsilon > 0$. By the definition of lim sup, there are only a finitely many n such that $\frac{1}{R'} + \varepsilon < |c_n|^{\frac{1}{n}}$. This gives us, $|c_n| > \left(\frac{1}{R'} + \varepsilon\right)^n$ for finitely many n. In other words,

$$|c_n| \leq \left(\frac{1}{R'} + \varepsilon\right)^n \quad \text{for all but finitely many } n$$

If $|z| < R'$, then there exists some ε such that, $|z| < \left(\frac{1}{R'} + \varepsilon\right)^{-1}$. In other words, $\left(\frac{1}{R'} + \varepsilon\right)|z| < 1$. Now, for all but finitely many n, we have

$$|c_n z^n| = |c_n|\,|z|^n \leq \left(\left(\frac{1}{R'} + \varepsilon\right)|z|\right)^n$$

As a result, the power series becomes is a convergent geometric series for all but finitely many n, so it converges.

Now, if $|z| > R'$, then $|z| = \left(\frac{1}{R'} - \varepsilon\right)^{-1}$ for some $\varepsilon > 0$. In a similar manner, one can now show that the power series is a geometric series for all but finitely many n. But this time, the common ratio is greater than 1. So the series diverges.

(b) Let $R' = \left(\liminf |c_n|^{\frac{1}{n}}\right)^{-1} = \liminf |c_n|^{-\frac{1}{n}} < \infty$. We shall prove that $r < R$ iff $r < R'$. That would result in $R' = R$. Assume $r < R'$. By the definition of $\liminf$,

$$r < R' = \liminf |c_n|^{-\frac{1}{n}} = \sup_{n\in\mathbb{N}} \left(\inf_{k\geq n} |c_k|^{-\frac{1}{k}}\right) \implies \exists N \in \mathbb{N} \text{ such that } r < \inf_{k\geq N} |c_k|^{-\frac{1}{k}}$$

Therefore, $r < |c_n|^{-\frac{1}{n}}$ for every $n \geq N$. So we have, for $n \geq N$,

$$r^n < |c_n|^{-1} \implies |c_n r^n| < 1 \implies (c_n r^n)_{n\in\mathbb{N}} \text{ is bounded } \implies r < R$$

Now let's assume $r < R$. So $c_n r^n$ is bounded, *i.e*, $|c_n r^n| < M$ for every n. Therefore,

$$|c_n r^n| < M \implies r^n < M\,|c_n|^{-1} \implies r < M^{\frac{1}{n}}\,|c_n|^{-\frac{1}{n}}$$

We know that, if $x < y_n$ for every n, then $x \leq \liminf y_n$ (the proof follows immediately from the definition of $\liminf$). As a result, $r \leq \liminf \left(M^{\frac{1}{n}}\,|c_n|^{-\frac{1}{n}}\right) = \liminf |c_n|^{-\frac{1}{n}} = R'$. So we are done!

(c) Assume that the limit exists and $R' = \lim\limits_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$. We wish to prove that $R' = R$. Again we assume WLOG that $z_0 = 0$.

Assume $|z| < R'$, so we can find r such that $|z| < r < R'$. As $R' = \lim \left|\frac{c_n}{c_{n+1}}\right|$, we can find an integer N such that $r < \left|\frac{c_n}{c_{n+1}}\right|$ for every $n \geq N$. Let $B = |c_N|\,r^N$. So

$$|c_{N+1}|\,r^{N+1} = |c_{N+1}|\,rr^N < |c_{N+1}| \left|\frac{c_N}{c_{N+1}}\right| r^N = B$$

$$\text{Similarly, } |c_{N+2}|\,r^{N+2} = |c_{N+2}|\,rr^{N+1} < |c_{N+2}| \left|\frac{c_{N+1}}{c_{N+2}}\right| r^{N+1} = |c_{N+1}|\,r^{N+1} < B$$

Inductively, $|c_n|\,r^n \leq B$ for every $n \geq N$. Now, for every $n \geq N$, we have

$$|c_n z^n| = |c_n|\,r^n \left(\frac{|z|}{r}\right)^n \leq B \left(\frac{|z|}{r}\right)^n$$

So the series is bounded a convergent geometric series for all but finitely many n. Therefore, the power series converges whenever $|z| < R'$.

Now assume $|z| > R'$, so we can find r such that $|z| > r > R'$. As $R' = \lim \left|\frac{c_n}{c_{n+1}}\right|$, we can find an integer N such that $r > \left|\frac{c_n}{c_{n+1}}\right|$ for every $n \geq N$. In a similar way as before, $|c_n|\,r^n \geq B = |c_N|\,r^N$ for every $n \geq N$. This gives us

$$|c_n z^n| = |c_n|\,r^n \left(\frac{|z|}{r}\right)^n \geq B \left(\frac{|z|}{r}\right)^n$$

$\frac{|z|}{r} > 1$, so this is not bounded. Therefore, the power series does not converge for $|z| > R'$. Hence we are done!

■

If you notice carefully, we are talking about convergence in $|z| < R$ and divergence in $|z| > R$. But what happens at the boundary, *i.e*, at $|z| = R$? The answer is we don't know. In fact, no one knows. Anything can happen at the boundary. Let's consider the following three examples:

$$\sum_{n=0}^{\infty} z^n \ , \quad \sum_{n=0}^{\infty} \frac{z^n}{n} \ , \quad \sum_{n=0}^{\infty} \frac{z^n}{n^2}$$

It's easy to check that all of them have radius of convergence 1. The first one converges at no point of the boundary; the third one converges at every boundary point. But the second one only converges at one boundary point and diverges at other.

§4.2 Differentiating Power Series

In an intuitive sense, it's kinda clear that differentiating $\sum c_n z^n$ "should" yield $\sum n c_n z^{n-1}$. We shall now proceed to prove this.

Lemma 4.2.1

The power series $\sum c_n z^n$ and $\sum n c_n z^{n-1}$ have the same radius of convergence.

Proof. We first prove that if $\sum |c_n z^n|$ converges for $|z| < R$, then $\sum |n c_n z^{n-1}|$ also converges for $|z| < R$. Similar as before, we choose ρ such that $|z| < \rho < R$, and we assume $z \neq 0$. Then we have

$$|n c_n z^n| = \frac{n}{|z|} \left(\frac{|z|}{\rho} \right)^n |c_n| \, \rho^n$$

Since $\frac{|z|}{\rho} < 1$, the series $\sum n \left(\frac{|z|}{\rho} \right)^n$ converges[1]. Thus there exists M such that $n \left(\frac{|z|}{\rho} \right)^n < M$ for every n. Hence,

$$|n c_n z^n| \leq \frac{M}{|z|} |c_n| \, \rho^n$$

$c_n \rho^n$ converges as $\rho < R$, so $\sum |n c_n z^{n-1}|$ also converges for $|z| < R$.

Now, conversely assume that $\sum |n c_n z^{n-1}|$ converges for $|z| < R$. Then

$$|c_n z^n| \leq |z| \left| n c_n z^{n-1} \right|$$

As $|c_n z^n|$ is bounded by a convergent series, so it is also convergent.

Therefore, $\sum c_n z^n$ and $\sum n c_n z^{n-1}$ have the same radius of convergence. ■

Proposition 4.2.2

Suppose that the power series $\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence $R > 0$. Then the function $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is differentiable on $D(0, R)$, with the derivative

$$f'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} z^n$$

[1] In fact, $\sum n r^n = \frac{r}{(1-r)^2}$ for $-1 < r < 1$

Proof. For $|z| < R$, we define $g(z) = \sum_{n=1}^{\infty} nc_n z^{n-1}$. Take any $w \in D(0,R)$. We claim that $f'(w)$ exists and $f'(w) = g(w)$. We define $s_n(z)$ and $R_n(z)$ as follows:

$$s_n(z) = \sum_{k=0}^{n} c_k z^k \ , \ R_n(z) = \sum_{k=n+1}^{\infty} c_k z^k \implies s_n'(z) = \sum_{k=1}^{n} kc_k z^{k-1} \implies \lim_{n\to\infty} s_n'(z) = g(z)$$

Therefore, $f(z) = s_n(z) + R_n(z)$. Since $|w| < R$, we can choose r such that $|w| < r < R$. Also, we can pick $\delta > 0$ such that $\bar{D}(w,\delta) \subseteq D(0,R)$. Let's take $z \in D(w,\delta)$. Let $\varepsilon > 0$.

$$\begin{aligned} \frac{f(z)-f(w)}{z-w} - g(w) &= \frac{s_n(z)+R_n(z)-s_n(w)-R_n(w)}{z-w} - g(w) \\ &= \frac{s_n(z)-s_n(w)}{z-w} - s_n'(w) + s_n'(w) - g(w) + \frac{R_n(z)-R_n(w)}{z-w} \end{aligned}$$

$$\left|\frac{f(z)-f(w)}{z-w} - g(w)\right| \le \left|\frac{s_n(z)-s_n(w)}{z-w} - s_n'(w)\right| + |s_n'(w)-g(w)| + \left|\frac{R_n(z)-R_n(w)}{z-w}\right|$$

Since $\lim_{n\to\infty} s_n'(z) = g(z)$, there exists $N_1 \in \mathbb{N}$ such that for every $n \ge N_1$, $|s_n'(w) - g(w)| < \frac{\varepsilon}{3}$. Since s_n is differetiable, we can choose z such that $\left|\frac{s_n(z)-s_n(w)}{z-w} - s_n'(w)\right| < \frac{\varepsilon}{3}$. Now,

$$\frac{R_n(z)-R_n(w)}{z-w} = \frac{1}{z-w}\sum_{k=n+1}^{\infty} c_k\left(z^k - w^k\right) = \sum_{k=n+1}^{\infty} c_k \frac{z^k-w^k}{z-w}$$

$$\begin{aligned} \therefore \left|\frac{R_n(z)-R_n(w)}{z-w}\right| &\le \sum_{k=n+1}^{\infty} |c_k| \left|\frac{z^k-w^k}{z-w}\right| \\ &= \sum_{k=n+1}^{\infty} |c_k| \left|z^{k-1} + z^{k-2}w + \cdots + w^{k-1}\right| \le \sum_{k=n+1}^{\infty} |c_k|\, kr^{k-1} \end{aligned}$$

Since $r < R$, the series $\sum |c_k|\, kr^{k-1}$ converges. So we can find $N_2 \in \mathbb{N}$ such that for every $n \ge N_2$, $\sum_{k=n+1}^{\infty} |c_k|\, kr^{k-1} < \frac{\varepsilon}{3}$. Putting all these together, for $n \ge \max(N_1, N_2)$,

$$\left|\frac{f(z)-f(w)}{z-w} - g(w)\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \implies \lim_{z\to w} \frac{f(z)-f(w)}{z-w} = g(w)$$

So we are done! ■

By this proposition, a power series is infinitely differentiable. Also, one can easily verify that

$$c_k = \frac{f^{(k)}(0)}{k!}$$

are the coefficients of the power series.

> **Definition 4.2.1** (Analytic Function). Let $V \subseteq C$ be open. A function $f : V \to \mathbb{C}$ is called analytic if for all $z_0 \in V$ there exists $r > 0$ such that there is a power series
>
> $$\sum_{n=0}^{\infty} c_n (z - z_0)^n$$
>
> converges to $f(z)$ on $D(z,r)$.

Proposition 4.2.2 basically tells us that analytic functions are holomorphic. The converse is also true, but that requires some more tools. We wish to prove it later.

5 Integration

Let $V \subseteq \mathbb{C}$ be open and $f : V \to \mathbb{C}$ be holomorphic. Suppose $\bar{D}(z_0, r) \subseteq V$ for some $z_0 \in V$ and $r > 0$. We will see that there exists a sequence $(c_n)_{n\in\mathbb{N}}$ such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \quad , \quad \forall\, z \in D(z, r)$$

How can we find these c_n? If we know that such c_n exists, then $c_n = \frac{f^{(n)}(z_0)}{n!}$ for every n. But what about their existence? We don't yet know that whether they always exist or not. In fact, they do exist, and they can be found using the formula

$$c_n = \frac{1}{2i\pi} \int_\gamma \frac{f(w)\; dw}{(w - z_0)^{n+1}} \quad , \quad \text{where } \gamma : [0, 2\pi] \to V \text{ is given by } \gamma(t) = z_0 + re^{it}$$

But what do these even mean? We shall get back to these after we build up how integrations work in $\mathbb{C}$.

§5.1 Contour Integral

Definition 5.1.1 (Curve/Path)**.** A **curve** or **path** is a continuous map γ from a closed interval $[a, b]$ to the complex plane $\mathbb{C}$. In other words, $\gamma : [a, b] \to \mathbb{C}$ is continuous. If $\gamma(a) = \gamma(b)$, we shall call it a **closed curve**.

We shall always assume γ is piecewise C^1 (continuous differentiable).

Definition 5.1.2 (Piecewise C^1 Curve)**.** A piecewise C^1 curve in $\mathbb{C}$ is a curve $\gamma : [a, b] \to \mathbb{C}$, which is continuously differentiable except at finitely many exceptional points.

Also, the range of γ is also called the trace of γ; it is often denoted by $\gamma^* = \gamma([a, b])$.

Abuse of Notation. People often write γ when they are actually refering to γ^*. For example, one may say something like, "the curve $\gamma \subseteq V$". But actually they mean $\gamma^* \subseteq V$.

For a function $f : [a, b] \to \mathbb{C}$, we can define its integral as the integral of its real and imaginary parts. Let $u(t) = \operatorname{Re} f(t)$ and $v(t) = \operatorname{Im} f(t)$.

$$\int_a^b f(t)\; dt = \int_a^b u(t)\; dt + i \int_a^b v(t)\; dt$$

Now we are ready to define contour integration.

Definition 5.1.3. Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise C^1 curve and $f : \gamma^* \to \mathbb{C}$ be a continuous function. The contour integral of f at γ is defined by

$$\int_\gamma f(z)\ dz := \int_a^b f(\gamma(t))\,\gamma'(t)\ dt$$

The integral on the RHS of the equation is our favorite Riemann integral. The map $t \mapsto f(\gamma(t))\,\gamma'(t)$ is a map from $[a,b]$ to $\mathbb{C}$. So its integral can be splitted into real and imaginary parts.

Example 5.1.1

Let $\gamma : [0, 2\pi] \to \mathbb{C}$ is given by $\gamma(t) = re^{it}$, for some fixed $r > 0$. Then

$$\int_\gamma z^n\ dz = \begin{cases} 2i\pi & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

The proof is pretty straightforward. Here $f(z) = z^n$. Let's do the $n = -1$ case first.

$$\int_\gamma z^{-1}\ dz = \int_0^{2\pi} \left(re^{it}\right)^{-1} ire^{it}\ dt = \int_0^{2\pi} i\ dt = 2i\pi$$

Now let's do the $n \neq -1$ case.

$$\begin{aligned} \int_\gamma z^n\ dz &= \int_0^{2\pi} \left(re^{it}\right)^n ire^{it}\ dt = ir^{n+1} \int_0^{2\pi} e^{i(n+1)t}\ dt \\ &= ir^{n+1} \int_0^{2\pi} \left(\cos\left((n+1)t\right) + i\sin\left((n+1)t\right)\right)\ dt = 0 \end{aligned}$$

The final integral is 0 because both $\cos\left((n+1)t\right)$ and $\sin\left((n+1)t\right)$ has 0 integral on the interval $[0, 2\pi]$ (left as an exercise for the reader).

Definition 5.1.4 (Primitive/Antiderivative)**.** Let $V \subseteq \mathbb{C}$ be open, and $f : V \to \mathbb{C}$ be continuous. Then f is said to have **antiderivative** or **primitive** on V if there exists a holomorphic function $F : V \to \mathbb{C}$ and $f = F'$ in V.

Proposition 5.1.1 (Fundamental Theorem of Calculus)

$V \subseteq \mathbb{C}$ is open $f : V \to \mathbb{C}$. Let $\gamma : [a,b] \to V$ with $\gamma(a) = z_1$ and $\gamma(b) = z_2$. If f has an antiderivative F on V, then

$$\int_\gamma f(z)\ dz = F(z_2) - F(z_1)$$

Proof. We shall use FTC for real variables and Chain Rule here.

$$\int_\gamma f(z)\ dz = \int_a^b f(\gamma(t))\,\gamma'(t)\ dt = \int_a^b F'(\gamma(t))\,\gamma'(t)\ dt = \int_a^b (F \circ \gamma)'(t)\ dt$$

Let $F \circ \gamma = g$. So g is a function from $[a,b]$ to $\mathbb{C}$. Let $g(t) = u(t) + iv(t)$, where $u(t), v(t)$

are functions from $[a,b]$ to $\mathbb{R}$. Then we have, $g'(t) = u'(t) + iv'(t)$. Therefore,

$$\begin{aligned}
&\int_a^b g'(t)\ dt = \int_a^b u'(t)\ dt + i\int_a^b v'(t)\ dt \\
\implies &\int_a^b g'(t)\ dt = u(b) - u(a) + iv(b) - iv(a) = g(b) - g(a) \\
\implies &\int_a^b (F\circ\gamma)'(t)\ dt = F(\gamma(b)) - F(\gamma(a)) = F(z_2) - F(z_1) \\
\implies &\boxed{\int_\gamma f(z)\ dz = F(z_2) - F(z_1)}
\end{aligned}$$

■

Corollary 5.1.2 (Cauchy's Theorem for Derivatives)
$V \subseteq \mathbb{C}$ is open $f : V \to \mathbb{C}$. Let $\gamma : [a,b] \to V$ be a closed curve on V. If f has an antiderivative F on V, then $\int_\gamma f(z)\ dz = 0$.

Proof. This is just FTC, but here we have $z_2 = z_1$ as γ is closed. ■

Since $F' = f$ on V, FTC can be rephrased as following:

$$\int_\gamma F'(z)\ dz = F(z_2) - F(z_1)$$

Lemma 5.1.3
Let $f : V \to \mathbb{C}$ where V is a connected open subset of $\mathbb{C}$. If $f' \equiv 0$ on V, then f is constant

Proof 1. The first proof follows directly from FTC. Let $z_1, z_2 \in V$ and $\gamma : [a,b] \to \mathbb{C}$ be a path connecting them. Then,

$$0 = \int_\gamma f'(z)\ dz = f(z_2) - f(z_1) \implies f(z_1) = f(z_2)$$

Therefore, f is constant. ■

Proof 2. This proof uses the fact that if X is connected and $A \subseteq X$ is both open and closed, then $A = \varnothing$ or $A = X$.
Let's fix $z_0 \in V$, and take $w_0 = f(z_0)$. Consider the set $A = \{z \in V\ :\ f(z) = w_0\}$. A is obviously nonempty. We wish to prove that A is both open and closed, that would imply $A = V$. We choose a sequence $\{z_n\}_{n\in N} \subseteq A$ with $z = \lim z_n$. Since $z_n \in A$, $f(z_n) = w_0$ for every n. As f is continuous,

$$w_0 = \lim_{n\to\infty} w_0 = \lim_{n\to\infty} f(z_n) = f\left(\lim_{n\to\infty} z_n\right) = f(z) \implies z \in A$$

So A is closed. Now we will show that A is open. We take $z_1 \in A$. We choose $\varepsilon > 0$ such that $B(z_1, \varepsilon) \subseteq V$. Such ε always exists because V is open. Take $z \in B(z_1, \varepsilon)$. We define $g : [0,1] \to \mathbb{C}$ as $g(t) = f(tz + (1-t)z_1)$. Then by Chain Rule,

$$g'(t) = f'(tz + (1-t)z_1)(z - z_1) = 0$$

g can be splitted into real part and imaginary part. Both real part and imaginary part has 0 derivative, and they are real valued functions. So g is constant. Therefore,

$$f(z) = g(1) = g(0) = f(z_1) = w_0 \implies z \in A \implies B(z_1, \varepsilon) \subseteq A$$

So A is open and we are done. ■

Lemma 5.1.4 (ML Inequality)

Suppose $\gamma : [a, b] \to \mathbb{C}$ is a smooth curve, and f is continuous on γ^*. If $|f| \leq M$ on γ^* and L is the length of γ, then

$$\left| \int_\gamma f(z)\ dz \right| \leq ML$$

Proof. The length of γ is given by

$$L = \int_a^b |\gamma'(t)|\ dt$$

Using the definition of integral,

$$\begin{aligned}\left| \int_\gamma f(z)\ dz \right| &= \left| \int_a^b f(\gamma(t))\gamma'(t)\ dt \right| \\ &\leq \int_a^b |f(\gamma(t))\gamma'(t)\ dt| \\ &\leq M \int_a^b |\gamma'(t)|\ dt = ML\end{aligned}$$

■

ML inequality is just a fancier name of triangle inequality. Because, if $f : [a, b] \to \mathbb{C}$, then triangle inequality gives us

$$\left| \int_a^b f(t)\ dt \right| \leq \int_a^b |f(t)|\ dt$$

But if we are integrating about some contour that is not a part of the real line, then triangle inequality basically says

$$\left| \int_\gamma f(z)\ dz \right| \leq \int_\gamma |f(z)|\ |dz|$$

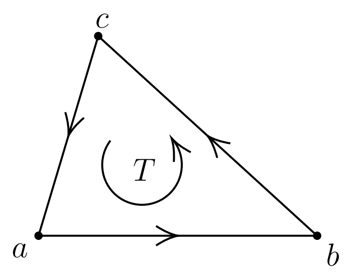
This is essentially the same as ML Inequality.

§5.2 Goursat's Theorem

Let T is be a (solid oriented) triangle with the vertices a, b, c (counterclockwise oriented), we write $T = [a, b, c]$. Then the boundary ∂T of T is the line segments that enclose T. In other words,

$$\partial T = [a, b] + [b, c] + [c, a]$$

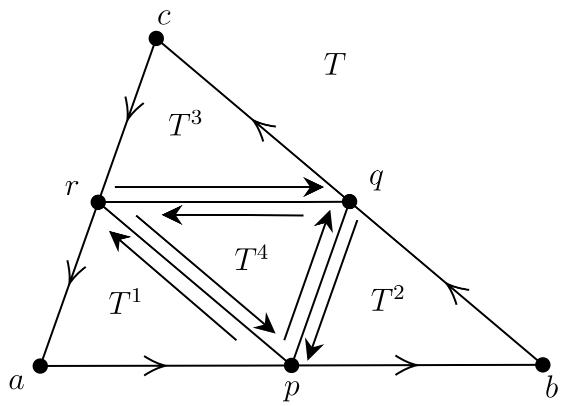
Here $[a, b]$ means the directed line segment from a to b.



Theorem 5.2.1 (Goursat's Theorem)
Let $V \subseteq \mathbb{C}$ be open, and $f : V \to \mathbb{C}$ be a holomorphic function. If T is a (solid oriented) triangle in V,

$$\int_{\partial T} f(z)\ dz = 0$$

Proof. We split T into four congruent triangles T^1, T^2, T^3, T^4.



$$\partial T = \partial T^1 + \partial T^2 + \partial T^3 + \partial T^4$$

Then we choose k such that

$$\left|\int_{\partial T^k} f(z)\ dz\right| = \max_{1\le j\le 4}\left|\int_{\partial T^j} f(z)\ dz\right|$$

and we define T_1 to be T^k. Then we have

$$\left|\int_{\partial T} f(z)\ dz\right| = \left|\sum_{j=1}^{4}\int_{\partial T^j} f(z)\ dz\right| \le \sum_{j=1}^{4}\left|\int_{\partial T^j} f(z)\ dz\right| \le 4\left|\int_{\partial T_1} f(z)\ dz\right|$$

Now we repeat the same argument for T_1 instead of T. Similar as before, we choose k such that

$$\left|\int_{\partial T_1^k} f(z)\ dz\right| = \max_{1\le j\le 4}\left|\int_{\partial T_1^j} f(z)\ dz\right|$$

and we define T_2 to be T_1^k. Then we have

$$\left|\int_{\partial T_1} f(z)\ dz\right| \le 4\left|\int_{\partial T_2} f(z)\ dz\right| \implies \left|\int_{\partial T} f(z)\ dz\right| \le 4^2\left|\int_{\partial T_2} f(z)\ dz\right|$$

If we keep continuing this process, T_{n+1} is the triangle among $T_n^1, T_n^2, T_n^3, T_n^4$ that gives the maximum integral along the boundary. Then, by induction, we have

$$\left|\int_{\partial T} f(z)\ dz\right| \le 4^n\left|\int_{\partial T_n} f(z)\ dz\right|$$

If $S \subseteq \mathbb{C}$, we define its diameter to be

$$\operatorname{diam}(S) = \sup_{z_1, z_2 \in S} |z_1 - z_2|$$

Then by the construction of T_1, $\operatorname{diam}(T_1) = \frac{1}{2}\operatorname{diam}(T)$. Inductively,

$$\operatorname{diam}(T_n) = 2^{-n}\operatorname{diam}(T)$$

The triangles T_n are closed. Also, $T \supseteq T_1 \supseteq T_2 \supseteq \cdots$, so the triangles are nested. Since $\operatorname{diam}(T_n) \to 0$, by the Nested Interval Theorem, there exists a unique point $z_0 \in \mathbb{C}$ such that

$$\{z_0\} = \bigcap_{n=1}^{\infty} T_n$$

f is holomorphic on V, so f is differentiable at z_0. Let $\alpha = f(z_0)$ and $\beta = f'(z_0)$. Then

$$f(z) = \alpha + \beta(z - z_0) + E(z),$$

where the error term $E(z)$ is a continuous function on V satisfying

$$\lim_{z \to z_0} \frac{E(z)}{|z - z_0|} = 0.$$

Now, $\alpha + \beta(z - z_0)$ is a polynomial, and thus it has an antiderivative $\alpha(z - z_0) + \beta(z - z_0)^2$. By Cauchy's Theorem for Derivatives, $\int_{\partial T_n} \alpha + \beta(z - z_0)\ dz = 0$. Therefore, for each n,

$$\left|\int_{\partial T} f(z)\ dz\right| \leq 4^n \left|\int_{\partial T_n} f(z)\ dz\right| \leq 4^n \left|\int_{\partial T_n} E(z)\ dz\right|$$

For each n, let M_n be the supremum of $|E(z)|$ on ∂T_n. If l is the length of ∂T, then the length of ∂T_n is $2^{-n} l$. By ML Inequality, we have

$$\left|\int_{\partial T_n} E(z)\ dz\right| \leq 2^{-n} l M_n \implies \left|\int_{\partial T} f(z)\ dz\right| \leq 2^n M_n l$$

Now we want to show that $2^n M_n \to 0$.

Since ∂T_n is compact and $E(z)$ is continuous, there exists $z_n \in T_n$ such that $|E(z_n)| = M_n$. Then

$$|z_n - z_0| \leq \operatorname{diam}(T_n) = 2^{-n}\operatorname{diam}(T) \to 0$$

So $z_n \to z_0$. Then

$$\lim_{z \to z_0} \frac{E(z)}{|z - z_0|} = 0 \implies \lim_{z \to z_0} \left|\frac{E(z)}{z - z_0}\right| = 0 \implies \lim_{n \to \infty} \left|\frac{E(z_n)}{z_n - z_0}\right| = 0$$

$$|2^n M_n| = |2^n E(z_n)| = 2^n \left|\frac{E(z_n)}{z_n - z_0}\right| |z_n - z_0| \leq 2^n \left|\frac{E(z_n)}{z_n - z_0}\right| 2^{-n}\operatorname{diam}(T)$$

$$\therefore \lim_{n \to \infty} |2^n M_n| \leq \operatorname{diam}(T) \lim_{n \to \infty} \left|\frac{E(z_n)}{z_n - z_0}\right| = 0$$

So $2^n m_n \to 0$. Since l is a constant,

$$\left|\int_{\partial T} f(z)\ dz\right| \leq 2^n M_n l \to 0 \implies \int_{\partial T} f(z)\ dz = 0$$

Hence, we are done! ■

Using Goursat's Theorem, one can conclude that if $f : V \to \mathbb{C}$ is holomorphic, then for any (solid oriented) polygon P in V,

$$\int_{\partial P} f(z)\ dz = 0$$

Because ∂P is the finite sum of some ∂T_i's where T_i's are triangles.

Proposition 5.2.2

Suppose V is a convex open subset of $\mathbb{C}$. Let $f : V \to \mathbb{C}$ be continuous and

$$\int_{\partial T} f(z)\ dz = 0\ , \quad \text{for any triangle } T \subseteq V.$$

Then f has an antiderivative on V.

Proof. Since V is convex, for every $a, b \in V$, $[a, b] \subseteq V$. We fix a $z_0 \in V$ and define F by

$$F(z) = \int_{[z_0,z]} f(w)\ dw$$

We claim that F is the desired primitive of f. We choose $z \in V$. Since V is open, we can find $r > 0$ such that $D(z, r) \subseteq V$. So we take $h \neq 0$ with $0 < |h| < r$, then $z + h \in V$. Consider the triangle $T = [z_0, z, z + h]$. Then $\partial T = [z_0, z] + [z, z + h] + [z + h, z_0]$.

$$0 = \int_{\partial T} f(w)\ dw = \int_{[z_0,z]} f(w)\ dw + \int_{[z,z+h]} f(w)\ dw + \int_{[z+h,z_0]} f(w)\ dw$$

$$\therefore \int_{[z,z+h]} f(w)\ dw = \int_{[z_0,z+h]} f(w)\ dw - \int_{[z_0,z]} f(w)\ dw = F(z+h) - F(z)$$

Claim — $\int_{[z,z+h]} dw = h$; also length of $[z, z + h]$ is $|h|$.

Proof. Here we are integrating $f_1(w) = 1$ over that path $[z, z + h]$. The path is $\gamma : [0, 1] \to V$, with $\gamma(t) = (1 - t)z + t(z + h)$; so $\gamma'(t) = -z + z + h = h$.

$$\int_{[z,z+h]} dw = \int_0^1 1 \cdot \gamma'(t)\ dt = \int_0^1 h\ dt = h\ , \text{ and } L = \int_0^1 |\gamma'(t)|\ dt = \int_0^1 |h|\ dt = |h|$$

□

Claim — $\lim_{h \to 0} \frac{1}{h} \int_{[z,z+h]} (f(w) - f(z))\ dw = 0$

Proof. Let $\varepsilon > 0$. Since f is continuous, there exists $\delta > 0$ such that whenever $|w - z| < \delta$, $|f(w) - f(z)| < \varepsilon$. We choose h with $|h| < \delta$. So if $w \in [z, z + h]$, we have $|z - w| \leq |h| < \delta$. As a result $|f(w) - f(z)| < \varepsilon$. By ML Inequality,

$$\left|\frac{1}{h} \int_{[z,z+h]} (f(w) - f(z))\ dw\right| = \left|\frac{1}{h}\right| \left|\int_{[z,z+h]} (f(w) - f(z))\ dw\right| \leq \left|\frac{1}{h}\right| \varepsilon\, |h| = \varepsilon$$

Taking $\varepsilon \to 0$, we are done. □

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{h}\int_{[z,z+h]}(f(w)-f(z))\ dw=0\\
\implies &\lim_{h\to 0}\frac{1}{h}\int_{[z,z+h]}f(w)\ dw=\lim_{h\to 0}\frac{1}{h}\int_{[z,z+h]}f(z)\ dw\\
\implies &\lim_{h\to 0}\frac{1}{h}\int_{[z,z+h]}f(w)\ dw=\lim_{h\to 0}\frac{f(z)}{h}\int_{[z,z+h]}dw\\
\implies &\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}=\lim_{h\to 0}\frac{f(z)}{h}h=f(z)\\
\implies &F'(z)=f(z)
\end{aligned}$$

Therefore $F'=f$ on V. So F is the desired antiderivative of f. ∎

Corollary 5.2.3

Let V be an open subset of $\mathbb{C}$ and $f:V\to\mathbb{C}$ be holomorphic. Then f has an antiderivative on every open convex subset of V.

Corollary 5.2.4 (Cauchy's Theorem in a Convex Set)

Let V be an open convex subset of $\mathbb{C}$ and $f:V\to\mathbb{C}$ be holomorphic. Then for any closed curve γ in V,

$$\int_\gamma f(z)\ dz=0$$

Proof. By Goursat's Theorem, for any triangle T in V,

$$\int_{\partial T} f(z)\ dz=0$$

V is convex, so by Proposition 5.2.2, f has an antiderivative on V. Then by Cauchy's Theorem for Derivatives, $\int_\gamma f(z)\ dz=0$. ∎

Definition 5.2.1 (Star Convex). A subset A of $\mathbb{R}^n$ is said to be **star convex** if for some point a_0 of A, all the line segments joining a_0 to other points of A lie in A.

Whenever we say "V is a **domain**", we actually mean that V is an open connected subset of $\mathbb{C}$.

Proposition 5.2.5 (Cauchy's Theorem in a Star-Convex Set)

Let V be a star-convex domain, and $f:V\to\mathbb{C}$ be a holomorphic function. Then

$$\int_\gamma f(z)\ dz=0$$

for any closed contour γ in V.

Proof. By Goursat's Theorem, for any triangle T in V,

$$\int_{\partial T} f(z)\ dz=0$$

Now, let z_0 be the point that is connected to every other point of V by a line segment. Then we define a new function F by

$$F(z) = \int_{[z_0,z]} f(w)\ dw$$

Mimicing the proof of Proposition 5.2.2, we get that F is a primitive of f. Then by Cauchy's Theorem for Derivatives, $\int_\gamma f(z)\ dz = 0$. ■

Proposition 5.2.6

Let $V \subseteq \mathbb{C}$ be a domain, and $f : V \to \mathbb{C}$ be a continuous function with the property

$$\int_\gamma f(z)\ dz = 0$$

for any closed contour γ in V. Then f has an antiderivative on V.

Proof. Firstly, note that the integral does not depend on the choice of path. This means, if $z_1, z_2 \in V$ and γ_1, γ_2 are two paths from z_1 to z_2, then

$$\int_{\gamma_1} f(z)\ dz = \int_{\gamma_2} f(z)\ dz$$

To prove this, notice that $\gamma_1 - \gamma_2$ is a closed loop based at z_1. Given that, the integral over any closed loop is 0. Hence,

$$\int_{\gamma_1-\gamma_2} f(z)\ dz = 0 \implies \int_{\gamma_1} f(z)\ dz - \int_{\gamma_2} f(z)\ dz = 0 \implies \int_{\gamma_1} f(z)\ dz = \int_{\gamma_2} f(z)\ dz$$

In other words, the integral does not depend on the choice of path. It only depends on the endpoints.

Now we shall just mimic the proof of Proposition 5.2.2. Fix $z_0 \in V$. For $z \in V$, take any path γ_z from z_0 to z, and define

$$F(z) = \int_{\gamma_z} f(w)\ dw$$

Then F is the antiderivative of f. The details are left for the reader to fill in. ■

6 CIF and Its Consequences

$D(z_0, r)$ means the open ball centered at z_0 with radius r. $\partial D(z_0, r)$ means the circle centered at z_0 with radius r. We give this circle counterclockwise orientation. So it can be expressed as the path $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = z_0 + re^{it}$. Also, we would use the shorthand symbol $\mathbb{D}$ for the unit open ball $D(0, 1)$.

§6.1 Cauchy Integral Formula

Lemma 6.1.1

Given $a, b \in \mathbb{C}$ and $z \in \mathbb{C}$, one can construct a continuous function $f : \mathbb{C} \to \mathbb{C}$ such that $f(z) = b$ and

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(w)}{w - z} \, dw = a$$

Proof. We define f as follows:

$$f(w) = (a - b)(w - z)|w| + b$$

Then obviously f is continuous and $f(z) = b$. Also, if $w \in \partial \mathbb{D}$, $|w| = 1$.

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(w)}{w - z} \, dw &= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{(a - b)(w - z)|w| + b}{w - z} \, dw \\
&= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} (a - b)|w| \, dw + \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{b}{w - z} \, dw \\
&= \frac{a - b}{2\pi i} \int_{\partial \mathbb{D}} dw + \frac{b}{2\pi i} \int_{\partial \mathbb{D}} \frac{dw}{w - z} \\
&= a - b + b = a
\end{aligned}$$

So this f is our desired continuous function. ■

Theorem 6.1.2 (Cauchy Integral Formula for Disks)

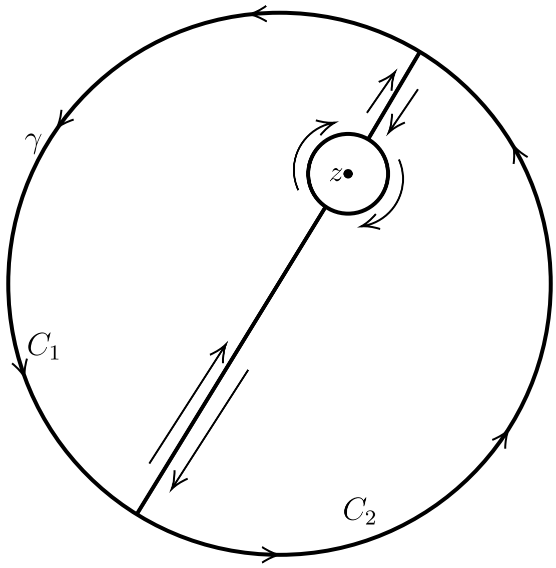
Let $V \subseteq \mathbb{C}$ be a domain and $f : V \to \mathbb{C}$ be holomorphic. Suppose $\overline{D(z_0, r)} \subseteq V$ for some $z_0 \in V$ and $r > 0$. Then for every $z \in D(z_0, r)$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(z_0, r)} \frac{f(w)}{w - z} \, dw.$$

Proof. Let γ be the circular path $\partial D(z_0, r)$ oriented counterclockwise. Take any point $z \in D(z_0, r)$. Then there exists $s > 0$ such that $D(z, s) \subseteq D(z_0, r)$. Let α be the circular path $\partial D(z, s)$ oriented counterclockwise. In other words,

$$\gamma(t) = z_0 + re^{it} \quad \text{and} \quad \alpha(t) = z + se^{it} \ , \quad \text{for } t \in [0, 2\pi]$$

Then we draw a diagonal line through the two centers, and it gives us two curves C_1 and C_2:



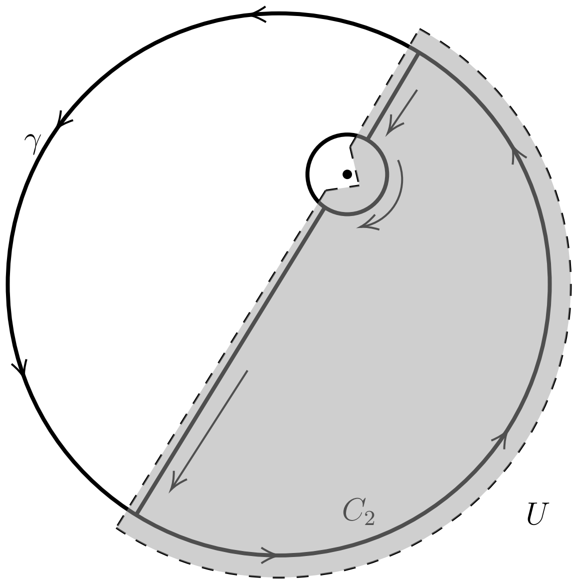
These curves have the property that

$$C_1 + C_2 = \gamma - \alpha \implies \gamma = C_1 + C_2 + \alpha$$

And $\frac{f(w)}{w-z}$ is holomorphic on the images of C_1, C_2, α, γ. Therefore,

$$\int_\gamma \frac{f(w)\ dw}{w-z} = \int_{C_1} \frac{f(w)\ dw}{w-z} + \int_{C_2} \frac{f(w)\ dw}{w-z} + \int_\alpha \frac{f(w)\ dw}{w-z}$$

Now, we can find a star convex, open set that contains C_1 and the space enclosed by C_1, and that star convex set does not contain z. Similarly, we can find one for C_2 also. An example for C_2 is given below:



Here $\frac{f(w)}{w-z}$ is holomorphic on U (because U does not contain z). U is star convex. So by Cauchy's Theorem in a Star-Convex Set,

$$\int_{C_2} \frac{f(w)\ dw}{w-z=0}$$

Similarly, $\int_{C_1} \frac{f(w)\ dw}{w-z} = 0$. Therefore,

$$\int_\gamma \frac{f(w)\ dw}{w-z} = \int_\alpha \frac{f(w)\ dw}{w-z}$$

Therefore,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma}\frac{f(w)}{w-z}\,dw &= \frac{1}{2\pi i}\int_{\gamma}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{0}^{2\pi}\frac{f\left(z+se^{it}\right)}{z+se^{it}-z}\,ise^{it}\,dt\\ &= \frac{1}{2\pi}\int_{0}^{2\pi} f\left(z+se^{it}\right)\,dt\end{aligned}$$

If we choose a even smaller positive value for s, it would stil be true.

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f(w)}{w-z}\,dw = \lim_{s\to 0^+}\frac{1}{2\pi}\int_{0}^{2\pi} f\left(z+se^{it}\right)\,dt$$

As $s\to 0^+$, $f\left(z+se^{it}\right)$ converges uniformly to $f(z)$. Therefore,

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi}\int_{0}^{2\pi} f(z)\,dt = f(z)$$

■

Example 6.1.1

Let's say we want to calculate the following integral:

$$\int_{\partial\mathbb{D}}\frac{\cos z^2+z}{z\left(z-\sqrt{\pi}\right)}\,dz$$

We denote this by I.

$$\frac{I}{2\pi i} = \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{\cos z^2+z}{z\left(z-\sqrt{\pi}\right)}\,dz = \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{\frac{\cos z^2+z}{z-\sqrt{\pi}}}{z-0}\,dz$$

$f(z) = \frac{\cos z^2+z}{z-\sqrt{\pi}}$ is holomorphic on $\overline{\mathbb{D}}$. So we can apply the Cauchy Integral Formula. Therefore,

$$\frac{I}{2\pi i} = f(0) = -\frac{1}{\sqrt{\pi}} \implies I = \frac{2\pi i}{\sqrt{\pi}}$$

Lemma 6.1.3 (Mean Value Property)

Let V be a domain and $f : V \to \mathbb{C}$ be holomorphic. Suppose $z_0 \in V$ with $\overline{D(z_0, r)} \subseteq V$. Then

$$f(z_0) = \frac{1}{2\pi}\int_{0}^{2\pi} f\left(z_0+re^{it}\right)\,dt$$

Proof. Let γ be the circular path $\partial D(z_0, r)$ oriented counterclockwise. In other words, $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then by Cauchy Integral Formula for Disks,

$$\begin{aligned}f(z_0) &= \frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{z-z_0}\,dz\\ &= \frac{1}{2\pi i}\int_{0}^{2\pi}\frac{f\left(z_0+re^{it}\right)}{z_0+re^{it}-z_0}\,ire^{it}\,dt\\ &= \frac{1}{2\pi}\int_{0}^{2\pi} f\left(z_0+re^{it}\right)\,dt\end{aligned}$$

■

Mean Value Property basically says that the value at the center of a circle is the average of the values at the boundary. There is a generalization of Mean Value Property, which says that the value at the center of a disk is the average of the values at the interior of the disk. See Problem 6 of Midterm 1.

Theorem 6.1.4 (Cauchy Integral Formula for Derivatives)
Let $V \subseteq \mathbb{C}$ be a domain and $f : V \to \mathbb{C}$ be holomorphic. Suppose $\overline{D(z_0, r)} \subseteq V$ for some $z_0 \in V$ and $r > 0$. Then for every $z \in D(z_0, r)$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(w)}{(w-z)^{n+1}}\, dw.$$

Proof. We shall approach by induction on n. The base case $n = 0$ is just Cauchy Integral Formula for Disks. Suppose inductively that f is $n-1$ times differentiable at z, and

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(w)}{(w-z)^{n}}\, dw$$

If $\frac{\partial}{\partial z} f^{(n-1)}(z)$ is a defined expression, it's equal to $f^{(n)}(z)$.

$$\begin{aligned} f^{(n)}(z) = \frac{\partial}{\partial z} f^{(n-1)}(z) &= \frac{\partial}{\partial z}\left(\frac{(n-1)!}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(w)}{(w-z)^{n}}\, dw\right) \\ &= \frac{(n-1)!}{2\pi i} \int_{\partial D(z_0,r)} f(w)\left(\frac{\partial}{\partial z}\frac{1}{(w-z)^{n}}\right) dw \\ &= \frac{n!}{2\pi i} \int_{\partial D(z_0,r)} f(w)\frac{1}{(w-z)^{n+1}}\, dw \end{aligned}$$

Here the integrand does not depend on z. That's why we could switch integral and $\frac{\partial}{\partial z}$. ■

One can conclude from this theorem that if f is once differentiable in V, then f is infinitely differentiable in V.

§6.2 Some Consequences

Theorem 6.2.1 (Morera's theorem)
Let $V \subseteq \mathbb{C}$ be a domain and suppose that $f : V \to \mathbb{C}$ is a continuous function. If for any closed curve γ in V

$$\int_\gamma f(z)\, dz = 0$$

then f is holomorphic.

Proof. For any closed curve γ the integral over γ is 0. So by Proposition 5.2.6, f has an antiderivative F in V. Since F is holomorphic, it is infinitely differentiable in V. So $F' = f$ is differentiable in V. So f is holomorphic. ■

Proposition 6.2.2

Holomorphic functions are analytic.

Proof. Let V be a domain and $f : V \to \mathbb{C}$ be holomorphic. Choose some $z_0 \in V$ and $r > 0$ such that $\overline{D(z_0, r)} \subseteq V$. Suppose $\gamma = \partial D(z_0, r)$ oriented counterclockwise. Then for $z \in D(z_0, r)$, Cauchy Integral Formula for Disks gives us

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)\ dw}{w-z} = \frac{1}{2\pi i}\int_\gamma \frac{f(w)\ dw}{(w-z_0)-(z-z_0)} \\ &= \frac{1}{2\pi i}\int_\gamma \frac{1}{w-z_0}\frac{1}{1-\frac{z-z_0}{w-z_0}} f(w)\ dw \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n\ dw \\ &= \sum_{n=0}^{\infty}(z-z_0)^n \frac{1}{2\pi i}\int_\gamma \frac{f(w)\ dw}{(w-z_0)^{n+1}} \\ &= \sum_{n=0}^{\infty} c_n (z-z_0)^n \ , \quad \text{where } c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)\ dw}{(w-z_0)^{n+1}} = \frac{f^{(n)}(z_0)}{n!} \end{aligned}$$

So f can be expressed as power series, hence it's analytic. ■

But there are a couple of issues with this proof. First of all, we need to make sure that $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ is a valid power series. In other words, it converges on $D(z_0, r)$. To make sure of that, we need the following cool result:

Lemma 6.2.3 (Cauchy Estimates)

Let V be a domain and $f : V \to \mathbb{C}$ is holomorphic. Let γ be the circle $|z - z_0| = r$ in V. Suppose $|f| \le M$ on γ^*. Then

$$\left|f^{(n)}(z_0)\right| \le \frac{Mn!}{r^n}$$

Proof. This follow immediately from Cauchy Integral Formula for Derivatives and ML Inequality. The length of γ is $2\pi r$.

$$\begin{aligned} \left|f^{(n)}(z_0)\right| &= \left|\frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\ dz\right| = \frac{n!}{2\pi}\left|\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\ dz\right| \\ &\le \frac{n!}{2\pi}\frac{M}{r^{n+1}}\ 2\pi r = \frac{Mn!}{r^n} \end{aligned}$$

■

Now, back to the proof of Proposition 6.2.2. Observe that,

$$|c_n (z-z_0)^n| = \left|\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n\right| \le M\left(\frac{|z-z_0|}{r}\right)^n$$

Clearly, $\sum_{n=0}^{\infty}\left(\frac{|z-z_0|}{r}\right)^n$ converges uniformly on $D(z_0, r)$. So $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ converges on $D(z_0, r)$.

Now, there is another issue with the proof of Proposition 6.2.2. We swapped summation and integral without any justification. There are a couple of ways to justify this. One can use either DCT or UCT. However, using DCT requires some amount of work.

One can apply *Weiestrass M-test* to show that the sum

$$\sum_{n=0}^{k} \frac{(z-z_0)^n}{(w-z_0)^{n+1}} f(w)$$

converges uniformly on γ^*. The argument is similar to the proof of Cauchy Estimates. Then UCT justifies swapping summation and integral.

Now our proof of Proposition 6.2.2 is complete. We previously showed that analytic functions are holomorphic. So, we have finally established the equivalence of holomorphicity and analyticity.

Theorem 6.2.4 (Liouville's Theorem)
If f is entire and bounded, then f is constant.

Proof. It's enough to show that $f' \equiv 0$. Let $z \in \mathbb{C}$. Choose any $r > 0$ and consider γ to be the circle centered at z with radius r. Then γ is contained in $\overline{D(z,r)}$. As a result, the length of γ is $2\pi r$. Since f is bounded, $|f(w)| \leq M$ for every w. By Cauchy integral formula for derivatives,

$$|f'(z)| = \left| \frac{1!}{2\pi i} \int_\gamma \frac{f(w)\, dw}{(w-z)^{1+1}} \right| \leq \frac{1}{2\pi} \frac{M}{r^2} 2\pi r = \frac{M}{r}$$

Since f is entire, this is true for every $r > 0$. So we can take $r \to \infty$ and we get $f'(z) = 0$. Therefore, $f' \equiv 0$ and f is constant. ■

Note that, Liouville's Theorem is a consequence of Cauchy Estimates. Now we will see a consequences of Liouville's theorem.

Theorem 6.2.5 (Fundamental Theorem of Algebra)
If $P \in \mathbb{C}[z]$ (P is a polynomial with complex coefficients), then P has a complex root.

Proof. Assume the contrary. Then $P(z) \neq 0$ for every $z \in \mathbb{C}$. So $f(z) = \frac{1}{P(z)}$ is entire.

Note that, as $|z| \to \infty$, $|P(z)| \to \infty$. So

$$\lim_{z\to\infty} f(z) = 0$$

Therefore, for $M > 0$ we can find $R > 0$ such that $|f(z)| < M$ when $|z| > R$. f is continuous on the compact set $\overline{D(0,R)}$. So it is bounded by some $N > 0$ in $\overline{D(0,R)}$. Therefore, f is bounded by $\max(M,N)$.

f is entire and bounded. So by Liouville's Theorem, f is constant. Therefore, $f(z) = 0$ for every z. But that's not possible since $f(z) = \frac{1}{P(z)}$. ■

Corollary 6.2.6
If $P \in \mathbb{C}[z]$, and the degree of P is n, then P has exactly n roots.

Proof. We shall proceed by induction. The 1-degree polynomials have exactly 1 root is trivial. Suppose P is a n-degree polynomial,

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

By FTA, P has at least one root w_1. Then writing $z = (z - w_1) + w_1$, plugging this into the expression of P, and using the binomial formula we get

$$P(z) = b_n (z - w_1)^n + b_{n-1} (z - w_1)^{n-1} + \cdots + b_1 (z - w_1) + b_0$$

where $b_0, b_1, \ldots, b_{n-1}, b_n$ are the new coefficients, with $b_n = a_n$. Since $P(w_1) = 0$, $b_0 = 0$. Therefore,

$$\begin{aligned} P(z) &= (z - w_1) \left(b_n (z - w_1)^{n-1} + b_{n-2} (z - w_1)^{n-1} + \cdots + b_2 (z - w_1) + b_1 \right) \\ &= (z - w_1) Q(z) \end{aligned}$$

Q is a $n-1$ degree polynomial. Therefore, by inductive hypothesis, it has exactly $n-1$ roots. Therefore, P has exactly n roots. So, by induction, we are done. ■

Lemma 6.2.7

Let $V \subseteq \mathbb{C}$ be a domain and $\overline{\mathbb{D}} \subseteq V$. Let $f : V \to \mathbb{C}$ be holomorphic, and

$$\int_{\partial \mathbb{D}} f(z) \overline{z}^n \, dz = 0$$

for every $n \geq 1$. Then $f \equiv 0$.

Proof. When $z \in \partial \mathbb{D}$,

$$1 = |z|^2 = z\overline{z} \implies \overline{z}^n = \frac{1}{z^n}$$

So, the integral condition becomes

$$\int_{\partial \mathbb{D}} \frac{f(z)}{z^n} \, dz = 0 \implies \frac{2\pi i}{(n-1)!} f^{(n-1)}(0)$$

If we consider power series around 0, we get that all the coefficients of the power series are 0. Hence, $f \equiv 0$. ■

§6.3 Some More Consequences

To avoid writing "$f : V \to \mathbb{C}$ is holomorphic" again and again, we shall write $f \in H(V)$ instead.

Proposition 6.3.1

Let $V \subseteq \mathbb{C}$ be a domain, and $f \in H(V)$. Suppose $f^{(n)}(z) = 0$ for every $n \in \mathbb{N}$ for some 0. Then f is a constant.

Proof. Our strategy is to show that $f'(z) = 0$ for every $z \in V$. Then by Lemma 5.1.3, we can conclude that f is a constant.

We take the following subset of V:

$$A = \left\{ z \in V \;:\; f^{(n)}(z) = 0 \;\; \forall n \in \mathbb{N} \right\}$$

A is clearly non-empty. To show that A is closed, suppose $(z_m)_{m\in\mathbb{N}}$ is a sequence in A that converges to z_0. Then by the continuity of $f^{(n)}$,

$$0 = \lim_{m\to\infty} f^{(n)}(z_m) = f^{(n)}\left(\lim_{m\to\infty} z_m\right) = f^{(n)}(z_0) \implies f^{(n)}(z_0) = 0 \implies z_0 \in A$$

Therefore, A is closed. Now we shall show that A is open.

Choose some $z_0 \in A$. Since V is open, there exists $r > 0$ such that $D(z_0, r) \subseteq V$. Then we take the power series around z_0. This gives us for $w \in D(z_0, r)$,

$$f(w) = \sum_{n=0}^{\infty} (w - z_0)^n \frac{f^{(n)}(z_0)}{n!} = f(z_0) + \sum_{n=1}^{\infty} (w - z_0)^n \frac{f^{(n)}(z_0)}{n!} = f(z_0)$$

So $f(w) = f(z_0)$ for every $w \in D(z_0, r)$. Therefore, $f^{(n)}(w) = 0$ for $n \in \mathbb{N}$. Hence $D(z_0, r) \subseteq A$, and thus A is open.

A is a nonempty clopen subset of the connected set V. Therefore, $A = V$. Therefore, $f'(z) = 0$ for all $z \in V$. So f is constant. ■

However, Prposition 6.3.1 is not true in real analysis. For example, if we take the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases},$$

f has all derivatives 0 at $x = 0$. But f is non-constant.

Definition 6.3.1 (Order of Zero)**.** Suppose f is holomorphic at 0 and $f(z) = 0$. If there exists a positive integer N with the property that

$$f^{(n)}(z) = 0 \quad \forall\, 0 \leq n < N \quad \text{and } f^{(N)}(z) \neq 0$$

Then we say that N is the order of zero at z. If there does not exists such positive integer N, we say that the order of zero at z is infinity. Otherwise, when $f(z) \neq 0$, we say that the order of zero at z is 0.

Corollary 6.3.2

Any zero of a nonconstant holomorphic function on a domain must be of finite order.

Proposition 6.3.3 (Factorization of Holomorphic Functions)

Let $f \in H(V)$ and $z_0 \in V$ be a zero of f with order n_0. Then there exists $g \in H(V)$ such that $g(z_0) \neq 0$ and

$$f(z) = (z - z_0)^{n_0} g(z), \qquad \forall\, z \in V$$

Proof. We choose $r > 0$ such that $D(z_0, r) \subseteq V$. We take the power series around z_0.

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

Since the zero at z_0 has order n_0, $c_n = 0$ for $0 \leq n < n_0$ and $c_{n_0} \neq 0$. Therefore,

$$f(z) = \sum_{n=n_0}^{\infty} c_n (z - z_0)^n = (z - z_0)^{n_0} \sum_{n=0}^{\infty} c_{n_0+n} (z - z_0)^n$$

This gives us that

$$\frac{f(z)}{(z - z_0)^{n_0}} = \sum_{n=0}^{\infty} c_{n_0+n} (z - z_0)^n$$

on $D(z_0, r) \cap (V \setminus \{z_0\})$. Now we define a function g that is $\frac{f(z)}{(z-z_0)^{n_0}}$ on $V \setminus \{z_0\}$, and $\sum_{n=0}^{\infty} c_{n_0+n} (z - z_0)^n$ on $D(z_0, r)$.

By *pasting lemma*, g is continuous. Also g is analytic on $D(z_0, r)$, so it is holomorphic on $D(z_0, r)$. Also, g is holomorphic on $V \setminus \{z_0\}$. So g is holomorphic. Furthermore,

$$g(z_0) = c_{n_0} \neq 0$$

So we are done! ■

Suppose $f \in H(V)$. Then the zero set of f, denoted by $Z(f)$, is defined by

$$Z(f) = \{z \in V \ : \ f(z) = 0\}$$

Theorem 6.3.4

If $V \subseteq \mathbb{C}$ is a domain and $f \in H(V)$ and f is non-constant, then $Z(f)$ does not contain any limit point.

Proof. Let $w \in V$ be a limit point of X, therefore it is the limit of some non-constant sequence in X. Let the sequence be $\{z_n\}_{n=1}^{\infty} \subseteq X$. Since f is continuous,

$$f(w) = f\left(\lim_{n\to\infty} z_n\right) = \lim_{n\to\infty} f(z_n) = 0 \implies w \in X$$

Since V is open, there exists $r > 0$ such that $D(w, r) \subseteq V$. We claim that $f \equiv 0$ on $D(w, r)$. Consider the power series of f around w.

$$f(z) = \sum_{n=0}^{\infty} c_n (z - w)^n$$

Assume for the sake of contradiction that f is not identically 0 on $D(w, r)$. So not all c_n's are 0. Let c_m be the first nonzero coefficient.

$$\begin{aligned} f(z) &= \sum_{n=m}^{\infty} c_n (z - w)^n = c_m (z - w)^m \left(1 + \sum_{n=m+1}^{\infty} \frac{c_n}{c_m} (z - w)^{n-m}\right) \\ &= c_m (z - w)^m (1 + g(z - w)) \end{aligned}$$

Notice that, if $z \to w$, then $g(z - w) \to 0$. Now take $z = z_n \neq w$.

$$f(z_n) = c_m (z_n - w)^m (1 + g(z_n - w))$$

$c_m (z_n - w)^m \neq 0$ as $z_n \neq w$. Also $z_n \to w$ gives us $1 + g(z_n - w) \to 1$. This means, for a suitable n, this quantity is arbitrarily close to 1. As a result, $1 + g(z_n - w) \neq 0$. So this gives us $f(z_n) \neq 0$, contradiction! Therfore, $D(w, r) \subseteq X$.

Let U be the interior of X. U contains $D(w, r)$, so it's nonempty. U is open because it's interior. Now we will show that U is also closed.

Let $a_n \in U$ with $a_n \to a$. By continuity, $f(a) = 0$. Then by the argument above, f is 0 in a neighborhood of a. Thus, $a \in U$, so U is closed. As U is both open and closed, and U is nonempty, so $U = V$. Therefore, $f(z) = 0$ for every $z \in V$. ■

Corollary 6.3.5

Let $V \subseteq \mathbb{C}$ be a domain and $f, g \in H(V)$. If $X = \{z \in V : f(z) = g(z)\}$ has a limit point in V, then $f \equiv g$ on V.

Proof. X is $Z(f-g)$. $Z(f-g)$ contains a limit point, so $f - g \equiv 0$. ■

Proposition 6.3.6 (Maximum Modulus Principle)

Let $V \subseteq \mathbb{C}$ be a domain, and $f \in H(V)$. If there exists $z_0 \in V$ such that $|f(z_0)| \geq |f(z)|$ for every $z \in V$, then f is constant.

Proof. We shall use Mean Value Property here. Choose $r > 0$ such that $\overline{D(z_0, r)} \subseteq V$. Then let γ be the path $\partial D(z_0, r)$ oriented counterclockwise. Then by Mean Value Property,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + re^{it}\right)\, dt \\ \therefore |f(z_0)| &= \frac{1}{2\pi} \left| \int_0^{2\pi} f\left(z_0 + re^{it}\right)\, dt \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(z_0 + re^{it}\right) \right|\, dt \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|\, dt = |f(z_0)| \end{aligned}$$

So all the inequality signs must be equality. Therefore,

$$\left| \int_0^{2\pi} f\left(z_0 + re^{it}\right)\, dt \right| = \int_0^{2\pi} \left| f\left(z_0 + re^{it}\right) \right|\, dt = \int_0^{2\pi} |f(z_0)|\, dt$$

Now, when does the first equality hold? It holds when $f(z_0 + re^{it})$ lies on the same ray from origin. In other words, when $f(z_0 + re^{it})$ has the same argument for all $t \in [0, 2\pi]$.

Also, when does the second equality hold? It holds when $|f(z_0 + re^{it})| = |f(z_0)|$. In other words, when $f(z_0 + re^{it})$ has the same modulus for all $t \in [0, 2\pi]$. Therefore, in order to satisfy the equalities, we must have $f(z_0 + re^{it})$ constant for $t \in [0, 2\pi]$. In other words, f is constant on γ^*.

$f(z_0)$ is the average of values at γ^*. Since f is constant on γ^*, $f(z) = f(z_0)$ for $z \in \gamma^*$. Now, consider $g(z) = f(z) - f(z_0)$. $g \in H(V)$ and $Z(g)$ contains the circle $\partial D(z_0, r)$. $\partial D(z_0, r)$ is closed, so it must have some limit points. Therefore, by Theorem 6.3.4, $g \equiv 0$. In other words, $f(z) = f(z_0)$ for every $z \in V$. ■

However, Maximum Modulus Principle is not true in real analysis. One simple counterexample is $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \cos x$. Then $|f(x)|$ has maximum at $x = 0$, but f is definitely non-constant.

Corollary 6.3.7 (Minimum Modulus Principle)
If f is a non-constant holomorphic function on a domain V, then there does not exist any $z_0 \in V$ such that $f(z_0) \neq 0$ and $|f(z_0)| \leq |f(z)|$ for every $z \in V$.

Proof. Assume the contrary. $f(z_0) \neq 0$ gives us $f(z) \neq 0$ for any $z \in V$. So $g(z) = \frac{1}{f(z)}$ is well-defined, and holomorphic on V. Then we have

$$|f(z_0)| \leq |f(z)| \implies \left|\frac{1}{f(z_0)}\right| \geq \left|\frac{1}{f(z)}\right| \implies |g(z_0)| \geq |g(z)|$$

Then by Maximum Modulus Principle, g is constant, which gives us that f is constant. Contradiction! ∎

Proposition 6.3.8
Let $V \subseteq \mathbb{C}$ be open, and $\phi : V \times [0,1]$ be continuous such that for each fixed $t \in [0,1]$, $z \mapsto \phi(z,t)$ is holomorphic. Then

$$g(z) = \int_0^1 \phi(z,t)\ dt$$

is a holomorphic function on V.

Proof. Let $T \subseteq V$ be a triangle in V. We claim that $\int_{\partial T} g = 0$.

$$\int_{\partial T} g(z)\ dz = \int_{\partial T} \int_0^1 \phi(z,t)\ dt\ dz = \int_0^1 dt \int_{\partial T} \phi(z,t)\ dz$$

Here we used *Fubini's theorem* to swap integrals. The integral over ∂T can be translated into an integral over a closed interval and the integrand is continuous. So we can apply *Fubini's theorem*. Since $\phi(z,t)$ is holomorphic for all t, by Goursat's Theorem,

$$\int_{\partial T} \phi(z,t)\ dz = 0$$

As a result,

$$\int_{\partial T} g(z)\ dz = \int_0^1 0\ dt = 0$$

So by Morera's theorem, g is holomorphic. ∎

Corollary 6.3.9
Suppose $V \subseteq \mathbb{C}$ is open, Γ is a chain, and $\phi : V \times \Gamma^*$ is a continuous function such that for each fixed $w \in \Gamma^*$, the function $z \mapsto \phi(z,w)$ is holomorphic. Then

$$g(z) = \int_\Gamma \phi(z,w)\ dw$$

is a holomorphic function on V.

The proof is exactly similar to that of Proposition 6.3.8.

§6.4 Homotopies and Simply Connected Domain

Definition 6.4.1 (Homotopy)**.** If f and f' are continuous maps of the space X into the space Y, we say that f is **homotopic** to f' if there is a continuous map $F : X \times I \to Y$ such that

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = f'(x)$$

for each $x \in X$. (Here $I = [0, 1]$) The map F is called a **homotopy** between f and f'. If f is homotopic to f', we write $f \simeq f'$.

We considered a continuous function $f : X \to Y$. Now we consider a special case where $f : [0, 1] \to X$ is a continuous map such that $f(0) = x_0 \in X$ and $f(1) = x_1 \in X$. We say that f is a path from x_0 (the initial point) to x_1 (the final point) in X.
If f and f' are 2 paths in X, there is a stronger relation between them than hometopy It is defined as follows:

Definition 6.4.2 (Path Homotopy)**.** Two paths $f, f' : I \to X$, are said to be **path homotopic** if they have the same initial and final points, dented by x_0 and x_1 respectively, and if there is a continuous map $F : I \times I \to X$ such that

$$\begin{aligned} F(s, 0) &= f(s) \quad \text{and} \quad F(s, 1) = f'(s), \\ F(0, t) &= x_0 \quad \text{and} \quad F(1, t) = x_1, \end{aligned}$$

for each $s \in I$ and each $t \in I$. We call F a **path homotopy** between f and f'. If f is path homotopic to f', we write $f \simeq_p f'$.

Lemma 6.4.1

Let (X, d) be a metric space, and $K \subseteq U \subseteq X$ where K is compact and U is open. Then there exists $\varepsilon > 0$ such that for every $k \in K$, $B(k, \varepsilon) \subseteq U$.

Proof. For every $x \in K$, since K is a subset of an open set U, there exists $r_x > 0$ such that $B(x, r_x) \subseteq U$. This gives us an open cover for K.

$$K \subseteq \bigcup_{x \in K} B\left(x, \frac{r_x}{2}\right)$$

Since K is compact, there is a finite subcover.

$$K \subseteq \bigcup_{i=1}^{n} B\left(x_i, \frac{r_{x_i}}{2}\right)$$

We take ε to be the smallest of these $\frac{r_{x_i}}{2}$.

$$\varepsilon = \min\left\{\frac{r_{x_i}}{2} \ : \ i = 1, 2, \ldots, n\right\}$$

We claim that this ε is our desired ε. Let $k \in K$. Take any $y \in B(k, \varepsilon)$. We need to show that $y \in U$.

$$y \in B(k, \varepsilon) \implies d(k, y) < \varepsilon \leq \frac{r_{x_i}}{2}$$

Suppose k is in $B\left(x_i, \frac{r_{x_i}}{2}\right)$.

$$k \in B\left(x_i, \frac{r_{x_i}}{2}\right) \implies d\left(x_i, k\right) < \frac{r_{x_i}}{2}$$

Using triangle inequality, we get

$$d\left(x_i, y\right) \leq d\left(x_i, k\right) + d\left(k, y\right) < r_{x_i} \implies y \in B\left(x_i, r_{x_i}\right) \subseteq U$$

Hence, $B\left(k, \varepsilon\right) \subseteq U$. ■

Proposition 6.4.2

Let $V \subseteq C$ be open in $\mathbb{C}$, and γ_1, γ_2 be two contours in V with $\gamma_1 \simeq_p \gamma_2$. Then

$$\int_{\gamma_1} f(z)\ dz = \int_{\gamma_2} f(z)\ dz$$

for any holomorphic function $f : V \to \mathbb{C}$.

Proof. $\gamma_1, \gamma_2 : I \to V$ are homotopic paths, with fixed endpoints; let $\gamma_1(0) = \gamma_2(0) = a$ and $\gamma_1(1) = \gamma_2(1) = b$. Let $F : I \times I \to V$ be a path homotopy between γ_1 and γ_2.

$$F(s, 0) = \gamma_1(s)\ ,\ F(s, 1) = \gamma_2(s) \ \text{ and }\ F(0, t) = a\ ,\ F(1, t) = b$$

for every $s, t \in I$. Now, $I \times I$ is a closed bounded subset of $\mathbb{R}^2$, so it's compact. Image of compact set under continuous map is compact, so $K = F(I \times I)$ is a compact subset of V. By Lemma 6.4.1, there exists $\varepsilon > 0$ such that for every $z \in K$, $D(z, \varepsilon) \subseteq V$.

Continuous function on a compact set is uniformly continuous (Heine-Cantor theorem). So F is uniformly continuous. Therefore, there exists $\delta > 0$ such that for $x, y \in I^2$

$$\text{whenever } \ \|x - y\| < \delta\ ,\ \text{we must have } \ |F(x) - F(y)| < \varepsilon$$

By the Archimedean property, there exists $N \in \mathbb{N}$ such that $N\delta > 1$. This can be rewritten as $\frac{1}{N} < \delta$. We divide I into N equal pieces, so we get

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = 1\ ,\ t_j = \frac{j}{N} \text{ for } j = 0, 1, \ldots, N$$

We apply this division to both dimensions of I^2. Thus we get N^2 squares of side length $\frac{1}{N}$, they make I^2. Also, we get $N + 1$ different paths from F by

$$\gamma_{t_j}(s) = F(s, t_j)$$

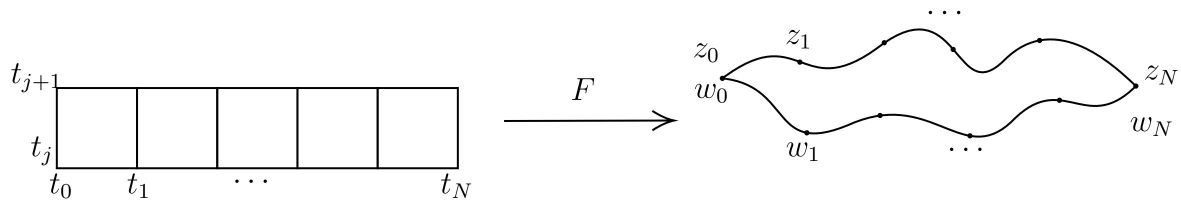
In this way, $\gamma_1 = \gamma_{t_0}$ and $\gamma_2 = \gamma_{t_N}$. We claim that for every $j = 0, 1, \ldots, N$

$$\int_{\gamma_{t_j}} f(z)\ dz = \int_{\gamma_{t_{j+1}}} f(z)\ dz$$

Let's consider the rectangular strip $I \times [t_j, t_{j+1}]$. By our previous construction, it is composed of N squares. Consider two sequences $w_0, w_1, \ldots, w_N$ and $z_0, z_1, \ldots, z_N$ defined by

$$w_i = \gamma_{t_j}(t_i) \quad \text{and} \quad z_i = \gamma_{t_{j+1}}(t_i)$$

Obviously, $w_0 = z_0$ and $w_N = z_N$ because they are the endpoints.



Let S_k denote the square $[t_{k-1}, t_k] \times [t_j, t_{j+1}]$ and c_k be the center of the square S_k. Let D_k be the open disk centered at $F(c_k)$ with radius ε, *i.e.* $D_k = D(F(c_k), \varepsilon)$. Using Lemma 6.4.1, $D_k \subseteq V$.

Claim — $z_k, z_{k-1}, w_k, w_{k-1} \in D_k$.

Proof. Let x be any of the corners of S_k. Since S_k has side length $\frac{1}{N}$,

$$\|c_k - x\| = \frac{1}{N}\frac{1}{\sqrt{2}} < \frac{1}{N} < \delta \implies |F(x) - F(c_k)| < \varepsilon \implies F(x) \in D_k$$

$S_k = [t_{k-1}, t_k] \times [t_j, t_{j+1}]$. If we take $x = (t_{k-1}, t_j)$,

$$D_k \ni F(x) = F(t_{k-1}, t_j) = \gamma_{t_j}(t_{k-1}) = w_{k-1}$$

If we take $x = (t_{k-1}, t_{j+1})$,

$$D_k \ni F(x) = F(t_{k-1}, t_{j+1}) = \gamma_{t_{j+1}}(t_{k-1}) = z_{k-1}$$

If we take $x = (t_k, t_j)$,

$$D_k \ni F(x) = F(t_k, t_j) = \gamma_{t_j}(t_k) = w_k$$

If we take $x = (t_k, t_{j+1})$,

$$D_k \ni F(x) = F(t_k, t_{j+1}) = \gamma_{t_{j+1}}(t_k) = z_k$$

The claim is proved. □

Now, D_k is a convex subset of V. So f has a primitive g_k on D_k. Therefore, both g_k and g_{k+1} are primitives of f on $D_k \cap D_{k+1}$. So $g_{k+1} - g_k$ must be a constant on $D_k \cap D_{k+1}$. By the above claim, $z_k, w_k \in (D_k \cap D_{k+1})$. Therefore,

$$g_{k+1}(z_k) - g_k(z_k) = g_{k+1}(w_k) - g_k(w_k) \implies \boxed{g_{k+1}(z_k) - g_{k+1}(w_k) = g_k(z_k) - g_k(w_k)}$$

for every $k = 1, 2, \ldots, N-1$. Now we can compute the integral using fundamental theorem of calculus.

$$\begin{aligned}\int_{\gamma_{t_{j+1}}} f(z)\,dz &= (g_1(z_1) - g_1(z_0)) + (g_2(z_2) - g_1(z_1)) + \cdots + (g_N(z_N) - g_N(z_{N-1})) \\ &= \sum_{k=1}^{N} (g_k(z_k) - g_k(z_{k-1}))\end{aligned}$$

We get a similar expression for $\int_{\gamma_{t_j}} f(z)\,dz$, with w_k instead of z_k. Now subtracting them, we get

$$\begin{aligned}\int_{\gamma_{t_{j+1}}} f(z)\ dz - \int_{\gamma_{t_j}} f(z)\ dz &= \sum_{k=1}^{N} \left(g_k\left(z_k\right) - g_k\left(z_{k-1}\right)\right) - \sum_{k=1}^{N} \left(g_k\left(w_k\right) - g_k\left(w_{k-1}\right)\right) \\ &= \sum_{k=1}^{N} \left(\left(g_k\left(z_k\right) - g_k\left(w_k\right)\right) - \left(g_k\left(z_{k-1}\right) - g_k\left(w_{k-1}\right)\right)\right) \\ &= \left(g_N\left(z_N\right) - g_N\left(w_N\right)\right) - \left(g_1\left(z_0\right) - g_1\left(w_0\right)\right) \\ &= 0\end{aligned}$$

Here we could do telescoping because of the boxed equation stated above. And the final quantity is 0 because $z_N = w_N$ and $z_0 = w_0$.

We have proved that $\int_{\gamma_{t_{j+1}}} f(z)\ dz = \int_{\gamma_{t_j}} f(z)\ dz$. Continuing this way, one can conclude that

$$\int_{\gamma_{t_0}} f(z)\ dz = \int_{\gamma_{t_N}} f(z)\ dz$$

γ_{t_0} is just γ_1 and γ_{t_N} is just γ_2. So we are done. ■

Definition 6.4.3 (Simply Connected Domain). A domain V is called **simply connected** if any two pair of curves in V with the same endpoints are homotopic.

Proposition 6.4.3

Any holomorphic function in a simply connected domain has a primitive.

Proof. Suppose $f \in H(V)$, where V is simply connected. From the simply connectedness, we get that the integral does not depend on the choice of path. It only depends on the endpoints.

Now we shall just mimic the proof of Proposition 5.2.2. Fix $z_0 \in V$. For $z \in V$, take any path γ_z from z_0 to z, and define

$$F(z) = \int_{\gamma_z} f(w)\ dw$$

Then F is the antiderivative of f. The details are left for the reader to fill in. ■

Corollary 6.4.4 (Cauchy's Theorem in a Simply Connected Domain)

Let V be a simply connected domain, and $f \in H(V)$. Then

$$\int_{\gamma} f(z)\ dz = 0$$

for any closed contour γ in V.

Proof. Proposition 6.4.3 tells us that f has an antiderivative. Then the conclusion follows from Cauchy's Theorem for Derivatives. ■

Simply connected set is usually a topological notion. But there are a couple of complex analytic characterization of simply connected sets.

(i) Suppose $V \subseteq \mathbb{C}$ is a domain with the property that for every $f \in H(V)$ and for all smooth closed curve γ in V,

$$\int_\gamma f(z)\ dz = 0\,.$$

Then V is simply connected.

(ii) If V is a domain where every holomorphic function has a holomorphic square root, then V is simply connected.

The proof of these two results are omitted.

7 Winding Numbers and Logarithms

In this chapter, we will try to answer the following question:

Question. Given an open set $V \subseteq \mathbb{C}$ and a *cycle* (a *cycle* is a chain composed of closed curves) Γ in V, when is it true that

$$\int_\Gamma f(z)\ dz = 0$$

for every $f \in H(V)$?

There is a similar question, which we can answer using the tools we have till now.

Question. Given an open set $V \subseteq \mathbb{C}$ and $f \in H(V)$, when is it true that

$$\int_\Gamma f(z)\ dz = 0$$

for all cycles Γ in V?

The answer to this question is provided by the following theorem:

Theorem 7.0.1

Suppose $V \subseteq \mathbb{C}$ is open and $f \in H(V)$. Then for every smooth curve γ in V

$$\int_\gamma f(z)\ dz = 0$$

if and only if f has a primitive.

Proof. One direction is given by Cauchy's Theorem for Derivatives. The other direction is Proposition 5.2.6. ■

In search for the primitive of $\frac{1}{z}$, we encounter complex logarithms.

§7.1 Logarithm

We say that w is a logarithm of z if

$$e^w = z\,.$$

Notice that, we said *a* logarithm, not *the* logarithm. That's because complex logarithm is not unique. For example, any complex number of the form $2\pi i n$ with $n \in \mathbb{Z}$ is a logarithm of 1. Furthermore,

$$e^w = z \implies w = \log|z| + \arg z\,.$$

But the problem is, arg is not a well-defined function. That's why we need the notion of *branch* of logarithm.

Definition 7.1.1 (Branch of Logarithm)**.** Let $V \subseteq \mathbb{C}$ be open. A **branch of logarithm** on V is a function $L \in H(V)$ such that

$$e^{L(z)} = z$$

for every $z \in V$.

One thing to note that, there certainly exists discontinuous functions f that satisfies $e^{f(z)} = z$. $f(z) = \log|z| + \operatorname{Arg} z$ is one such example. But we need not only continuous, but also holomorphic function to be a branch of logarithm. The following result tells us that continuity is enough.

Proposition 7.1.1

Suppose $V \subseteq \mathbb{C}$ is open and $f \in C(V)$ that satisfies $e^{f(z)} = z$ for every $z \in V$. Then $f \in H(V)$.

Proof. Firstly, note that f is injective. To prove that let $f(z) = f(w)$. Then

$$z = e^{f(z)} = e^{f(w)} = w$$

So f is injective. In particular, for every small nonzero h, $f(z+h) - f(z) \neq 0$. Therefore,

$$\lim_{h \to 0} \frac{h}{f(z+h) - f(z)} = \lim_{h \to 0} \frac{z+h-z}{f(z+h) - f(z)} = \lim_{h \to 0} \frac{e^{f(z+h)} - e^{f(z)}}{f(z+h) - f(z)} = e^{f(z)} = z$$
$$\therefore \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{1}{z}$$

Since $z = e^{f(z)}$, z is not equal to 0. Therefore, for $z \in V$, $f'(z)$ exists and it is equal to $\frac{1}{z}$. ∎

The following result gives us a characterization of branches of logarithm on a domain.

Proposition 7.1.2

Let $V \subseteq \mathbb{C}$ be a domain and $L \in H(V)$. Then the following are equivalent:

(i) L is a branch of logarithm in V.

(ii) $e^{L(z_0)} = z_0$ for some $z_0 \in V$, and $L'(z) = \frac{1}{z}$ for every $z \in V$.

Proof. (i)⇒(ii) is trivial, so let's prove (ii)⇒(i). We define a new function $f(z) = e^{-L(z)}z$. Then $f \in H(V)$. Then for any $z \in V$,

$$f'(z) = e^{-L(z)} - zL'(z)\, e^{-L(z)} = 0$$

So $f' \equiv 0$ on V. Therefore, by Lemma 5.1.3, f is constant. Then for any $z \in V$,

$$f(z) = f(z_0) = e^{-L(z_0)} z_0 = 1 \implies e^{L(z)} = z$$

Therefore, L is a branch of logarithm in V. ∎

Proposition 7.1.3

Let $V \subseteq \mathbb{C}$ be a domain. There exists a branch of logarithm in $V \iff$ there exists $f \in H(V)$ such that $f'(z) = \frac{1}{z}$ for all $z \in V$.

Proof. $\Rightarrow$ is trivial. Let's prove $\Leftarrow$. We choose some $z_0 \in V$. Since $z_0 \neq 0$, we can find $c \in \mathbb{C}$ such that

$$e^{f(z_0)+c} = z_0$$

We define $L(z) = f(z) + c$. Then $L'(z) = \frac{1}{z}$ and L is a logarithm at $z_0 \in V$. Therefore, by Proposition 7.1.2, L is a branch of logarithm. ∎

Corollary 7.1.4

If V is a convex open subset of $\mathbb{C} \setminus \{0\}$, then there exists a branch of logarithm in V.

Proof. V does not contain 0, so $z \mapsto \frac{1}{z}$ is holomorphic on V. V is convex, so by Corollary 5.2.3, $\frac{1}{z}$ has an antiderivative. So we can find $f \in H(V)$ with the property that $f'(z) = \frac{1}{z}$ for all $z \in V$. Hence, by Proposition 7.1.3, there exists a branch of logarithm in V. ∎

In fact, Corollary 7.1.4 still holds if we have a simply connected set instead of convex set. Because, Proposition 6.4.3 tells us that any holomorphic function on a simply connected domain has an antiderivative. So there is some $f \in H(V)$ with $f'(z) = \frac{1}{z}$. Hence, there exists a branch of logarithm.

Proposition 7.1.5

Suppose that V is an open subset of $\mathbb{C}$. There exists a brach of logarithm in V if and only if

$$\int_\gamma \frac{dz}{z} = 0$$

for all closed path γ in V.

Proof. Using the proof of Corollary 7.1.4, branch of logarithm exists if and only if there is a primitive of $\frac{1}{z}$. By Theorem 7.0.1, primitive of $\frac{1}{z}$ exists if and only if the integral of $\frac{1}{z}$ over any closed path is 0. ∎

§7.2 Winding Numbers

We've seen before that

$$\int_\gamma \frac{dz}{z} = 2\pi i \ , \qquad \text{where } \gamma = \partial\mathbb{D}$$

If we take $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = e^{-it}$, then we get that

$$\int_\gamma \frac{dz}{z} = -2\pi i$$

Furthermore, if we take $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = e^{2it}$, then the integral would be $4\pi i$. So we see that if γ revolves around 0 n-times (counterclockwise rotation is positive, and clockwise is negative), the integral of $\frac{1}{z}$ over γ is $2\pi ni$. Also, if we translate things, we can find how many times a curve revolves around any point. This gives us motivation to define **winding number** or the **index** of a curve. We can define it as follows:

$$\operatorname{Ind}(\gamma, a) = \frac{1}{2\pi i} \frac{dz}{z-a}$$

Then $\operatorname{Ind}(\gamma, a)$ will give us the number how many times γ winds around a. But this definition only works if we can integrate over γ. Whenever γ is not piecewise smooth, we can't integrate, so this definition of winding number doesn't work. So we need a better definition that works for continuous paths too. The following lemma allows us to do so.

Lemma 7.2.1

Let $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ be continuous. Suppose $\theta_0 \in \mathbb{R}$ such that $e^{i\theta_0} = \frac{\gamma(0)}{|\gamma(0)|}$. Then there exists a unique continuous function $\theta : [0,1] \to \mathbb{R}$ such that

$$\gamma(t) = |\gamma(t)| e^{i\theta(t)}$$

for all $t \in [0,1]$ and $\theta(0) = \theta_0$.

Proof. $\gamma(t) \neq 0$, so there exists $\epsilon > 0$ such that $D(0, \epsilon)$ does not intersect γ^*. $[0,1]$ is compact, so γ is uniformly continuous. Therefore, we can find $\delta > 0$ such that

$$|\gamma(s) - \gamma(t)| < \epsilon \quad \text{whenever} \quad |s-t| < \delta$$

By the Archimedian property, there exists $n \in \mathbb{N}$ such that $n\delta > 1$. Now we divide $[0,1]$ into n equal parts.

Let $I_j = \left[\frac{j-1}{n}, \frac{j}{n}\right]$. Then for $s, t \in I_j$,

$$|s-t| \leq \frac{1}{n} < \delta \implies |\gamma(s) - \gamma(t)| < \epsilon$$

So $\gamma(I_j)$ is contained in a disk D_j of radius ϵ not containing the origin. Since $0 \notin D_j$, $\frac{1}{z}$ is holomorphic on D_j. D_j is convex, so there is a branch of logarithm in D_j. In other words, there exists $L_j \in H(D_j)$ such that $e^{iL_j(z)} = z$ for all $z \in D_j$. Now, $\gamma(0) \in D_1$, so

$$e^{iL_1(\gamma(0))} = \gamma(0) = |\gamma(0)| e^{i\theta_0}$$

Therefore, $\operatorname{Re} L_1(\gamma(0))$ and θ_0 differ by an integer multiple of 2π.

Adding $2\pi i$ to a branch of logarithm yields another branch. So we can replace L_1 by $L_1 + 2\pi n_1$ in such a way that $\operatorname{Re} L_1(\gamma(0)) = \theta_0$. As a result,

$$\gamma(0) = e^{iL_1(\gamma(0))} = e^{i\theta_0} |\gamma(0)|$$

Now we shall approach inductively. Note that, for $j \geq 2$, $\gamma\left(\frac{j}{n}\right) \in D_j \cap D_{j+1}$. $L_j\left(\gamma\left(\frac{j}{n}\right)\right)$ and $L_{j+1}\left(\gamma\left(\frac{j}{n}\right)\right)$ exists.

$$e^{iL_j\left(\gamma\left(\frac{j}{n}\right)\right)} = \gamma\left(\frac{j}{n}\right) = e^{iL_{j+1}\left(\gamma\left(\frac{j}{n}\right)\right)}$$

So $\operatorname{Re} L_j\left(\gamma\left(\frac{j}{n}\right)\right)$ and $\operatorname{Re} L_{j+1}\left(\gamma\left(\frac{j}{n}\right)\right)$ differ by an integer multiple of 2π.

Adding $2\pi i$ to a branch of logarithm yields another branch. So we can replace L_{j+1} by $L_{j+1} + 2\pi n_{j+1}$ in such a way that $\operatorname{Re} L_j\left(\gamma\left(\frac{j}{n}\right)\right) = \operatorname{Re} L_{j+1}\left(\gamma\left(\frac{j}{n}\right)\right)$. Therefore,

$$L_j\left(\gamma\left(\frac{j}{n}\right)\right) = L_{j+1}\left(\gamma\left(\frac{j}{n}\right)\right)$$

for all $j \in \{1, 2, \ldots, n-1\}$.

Now we can just define $\theta(t) = \operatorname{Re} L_j(\gamma(t))$ if $t \in I_j$. Then this θ satisfies our condition $\gamma(t) = |\gamma(t)| e^{i\theta(t)}$. Also, θ is continuous on the closed pieces I_j, and agrees at the intersection of the pieces. Therefore, by *pasting lemma*, θ is continuous.

Now we are left with the uniqueness of θ. Suppose that there are two such continuous maps θ_1 and θ_2. Then we have

$$|\gamma(t)| e^{i\theta_1(t)} = \gamma(t) = |\gamma(t)| e^{i\theta_2(t)} \implies e^{i(\theta_1(t)-\theta_2(t))} = 1$$

So $\frac{\theta_1(t)-\theta_2(t)}{2\pi} \in \mathbb{Z}$. Now, $t \mapsto \frac{\theta_1(t)-\theta_2(t)}{2\pi}$ is a continuous map from $[0,1]$ to $\mathbb{Z}$. Since $[0,1]$ is connected, the map must be constant. As $\theta_1(0) = \theta_0 = \theta_2(0)$, $\theta_1(t) - \theta_2(t) = 0$ for all $t \in [0,1]$. Hence, the map θ is unique. ■

Now we are ready to define winding number.

Definition 7.2.1 (Winding Number). If $\gamma : [0,1] \mathbb{C}$ is a continuous closed curve, and $a \notin \gamma^*$, then the **winding number** or **index** of γ about a is

$$\operatorname{Ind}(\gamma, a) = \frac{\theta(1) - \theta(0)}{2\pi},$$

where $\theta : [0,1]$ is a continuous function such that

$$\gamma(t) - a = |\gamma(t) - a| e^{i\theta(t)}$$

Existence of such θ is guaranted by Lemma 7.2.1.

Now we will show that this definition agrees with our "idea" of winding numbers.

Lemma 7.2.2

If γ is a closed path, and θ is that unique continuous function of Lemma 7.2.1, then $\frac{\theta(1)-\theta(0)}{2\pi}$ is an integer.

Proof. Since γ is a closed path, $\gamma(1) = \gamma(0) = z_0$. Both $\theta(0)$ and $\theta(1)$ are arguments of z_0. Hence, these arguments differ by an integer multiple of 2π. That's why

$$\theta(1) - \theta(0) = 2\pi n, \quad \text{where } n \in \mathbb{Z} \implies \frac{\theta(1) - \theta(0)}{2\pi} = n \in \mathbb{Z}$$

■

Proposition 7.2.3

If γ is a smooth closed curve and $a \notin \gamma^*$, then

$$\operatorname{Ind}(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - a}$$

Proof. $\gamma : [0,1] \to \mathbb{C}$, so we can write

$$\gamma(t) = a + r(t) e^{i\theta(t)}$$

where $r(t) = |\gamma(t) - a| > 0$. Since $r \neq 0$, r is differentiable. Also, differentiablity of γ guarantees differentiability of θ.

$$\int_\gamma \frac{dz}{z-a} = \int_0^1 \frac{\gamma'(t)\ dt}{\gamma(t)-a} = \int_0^1 \frac{r'(t)\, e^{i\theta(t)} + ir(t)\, e^{i\theta(t)}\theta'(t)}{r(t)\, e^{i\theta(t)}}\ dt$$

$$= \int_0^1 \frac{r'(t)}{r(t)}\ dt + i\int_0^1 \theta'(t)\ \ dt = \ln(r(1)) - \ln(r(0)) + i(\theta(1) - \theta(0))$$

$$\therefore \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = \frac{\theta(1)-\theta(0)}{2\pi} = \operatorname{Ind}(\gamma, a)$$

■

Corollary 7.2.4

If γ_1, γ_2 are two homotopic smooth curves in $\mathbb{C} \setminus \{a\}$, then

$$\operatorname{Ind}(\gamma_1, a) = \operatorname{Ind}(\gamma_2, a)$$

Proof. Trivial from Proposition 7.2.3 and Proposition 6.4.2. ■

Proposition 7.2.5

Suppose V is an open subset of $\mathbb{C}$. There exists a branch of logarithm in V if and only if $0 \notin V$ and

$$\operatorname{Ind}(\gamma, 0) = 0$$

for any closed curve γ in V.

This is a restatement of Proposition 7.1.5. From this, we can conclude that there cannot be any branch of logarithm in the whole punctured plane $\mathbb{C} \setminus \{0\}$. But if you get rid of the negative real axis, then we have a branch of logarithm. In a similar manner, if you delete any ray from 0, then there exists a branch of logarithm in that set.

$$\text{There is a branch of logarithm in } \mathbb{C} \setminus \left\{re^{i\theta}\ :\ r \in [0, \infty)\right\}$$

This is not hard at all to prove. Because $S = \mathbb{C} \setminus \left\{re^{i\theta}\ :\ r \in [0, \infty)\right\}$ is a star convex set, so it's simply connected. So we have a branch of logarithm in S.

In fact, a stronger result is true. There are no other set S' that contains S and there is a branch of logarithm in S'. The proof is also pretty simple. Because if S' is a strictly larger set than S, it contains at least one point of the form $r_0 e^{i\theta}$. Then if we integrate $\frac{1}{z}$ over $\partial D(0, r_0)$, the integral is not 0.

§7.3 Branch of Logarithm of Function

Proposition 7.3.1

Let V be a simply connected subset of $\mathbb{C}$. If $f \in H(V)$ and f has no zeroes on V, then there exists $L \in H(V)$ such that

$$e^L = f \quad \text{in } V$$

Proof. f is nonzero, f' is holomorphic, therefore $\frac{f'}{f}$ is holomorphic. Since V is simply connected, by Proposition 6.4.3, $\frac{f'}{f}$ has a primitive F. Now fe^{-F} is holomorphic. Hence,

$$\left(fe^{-F}\right)' = f'e^{-F} + fe^{-F}\left(-F'\right) = f'e^{-F} - fe^{-F}\frac{f'}{f} = 0$$

So fe^{-F} is constant. Both f and e^{-F} are nonzero, so $fe^{-F} = e^{c} \neq 0$.

$$f\left(z\right)e^{-F(z)} = e^{c} \implies f\left(z\right) = e^{F(z)+c}$$

So $L = F + c$ is our desired holomorphic function. ■

Similar to logarithm, taking n-th root is also a problem in $\mathbb{C}$. Because there are n different values for the n-th root of 1 (known as the n-th roots of unity). That's why there is no well-defined function, in general, for taking n-th root. But things are so smooth when we have a simply connected set.

Proposition 7.3.2

Let V be a simply connected subset of $\mathbb{C}$ and $n \in \mathbb{N}$. If $f \in H\left(V\right)$ is nonvanishing, then there exists $g \in H\left(V\right)$ such that $g^n = f$.

Proof. f is nonvanishing holomorphic in a simply connected domain. Therefore, by Proposition 7.3.1, there exists $L \in H\left(V\right)$ such that

$$e^{L(z)} = f\left(z\right) \ , \quad \text{for every } z \in V$$

Now, $\frac{L(z)}{n}$ is a holomorphic function. So we can just take $g\left(z\right) = \exp\left(\frac{L(z)}{n}\right)$. Then we have

$$g\left(z\right)^n = \left(e^{\frac{L(z)}{n}}\right)^n = e^{L(z)} = f\left(z\right)$$

■

§7.4 Cauchy's Theorem

In this section we shall see some generalization of Cauchy's theorems for cycles.

Lemma 7.4.1

Suppose Γ is a cycle in $\mathbb{C}$. The function $I\left(a\right) = \operatorname{Ind}\left(\Gamma, a\right)$ is an integer valued continuous function on $\mathbb{C} \setminus \Gamma^*$.

Proof. For a closed curve γ, $\operatorname{Ind}\left(\gamma, a\right)$ is an integer. Since a cycle is a chain of closed curves,

$$\Gamma = \sum \gamma \implies \operatorname{Ind}\left(\Gamma, a\right) = \sum \operatorname{Ind}\left(\gamma, a\right) \in \mathbb{Z}$$

Now we need to show that I is a continuous function. Let's fix some $p_0 \in \mathbb{C} \setminus \Gamma^*$. Let $d = d\left(\Gamma^*, p_0\right)$ be the distance from p_0 to Γ^*. To put it concretely,

$$d\left(\Gamma^*, p_0\right) = \inf\left\{\left|z - p_0\right| \ : \ z \in \Gamma^*\right\} .$$

Since Γ^* is compact, $d > 0$. Therefore,

$$p \in D\left(p_0, \frac{d}{2}\right) \implies \left|z - p\right| \geq \left|z - p_0\right| - \left|p - p_0\right| \geq \frac{d}{2} \ , \quad \text{for } z \in \Gamma^*$$

Let $\varepsilon > 0$. We choose $\delta > 0$ in such a way that

$$\delta < \min\left\{\frac{d}{2}, \frac{\varepsilon\pi d^2}{l}\right\}$$

where l is the length of Γ. Now, for $|p - p_0| < \delta$,

$$\begin{aligned}|I(p) - I(p_0)| &= \left|\frac{1}{2\pi i}\int_\Gamma\left(\frac{1}{z-p} - \frac{1}{z-p_0}\right)dz\right| = \frac{1}{2\pi}\left|\int_\Gamma \frac{p_0 - p}{(z-p)(z-p_0)}\,dz\right| \\ &\le \frac{1}{2\pi}\int_\Gamma \frac{|p_0 - p|\ |dz|}{|z-p|\,|z-p_0|} \le \frac{1}{2\pi}\frac{2\delta l}{d^2} \\ &\le \frac{\delta l}{\pi d^2} < \varepsilon\end{aligned}$$

So I is continuous at p_0. Similarly, I is continuous at every point of $\mathbb{C}\setminus\Gamma^*$. ■

Corollary 7.4.2

$\operatorname{Ind}(\Gamma, a)$ is constant on the components of $\mathbb{C}\setminus\Gamma^*$, and vanished on the unbounded component.

Proof. The components are maximal connected subsets of $\mathbb{C}\setminus\Gamma^*$. Since $\operatorname{Ind}(\Gamma, a)$ is continuous integer valued, it's constant on the connected subsets. Hence, $\operatorname{Ind}(\Gamma, a)$ is constant on the components of $\mathbb{C}\setminus\Gamma^*$.

For any $p \in \mathbb{C}\setminus\Gamma^*$,

$$|\operatorname{Ind}(\Gamma, p)| = \frac{1}{2\pi}\left|\int_\Gamma \frac{dz}{z-p}\right| \le \frac{1}{2\pi}\int_\Gamma \frac{|dz|}{|z-p|} \le \frac{l}{2\pi\ d(\Gamma^*, p)}$$

If p is in the unbounded component, $d(\Gamma^*, p)$ can be arbitrarily large. So $\operatorname{Ind}(\Gamma, p)$ is bounded by arbitrarily small numbers. Therefore, $\operatorname{Ind}(\Gamma, p) = 0$. ■

Lemma 7.4.3

Let $V \subseteq \mathbb{C}$ be open and $f \in H(V)$. We define $g : V \times V \to \mathbb{C}$ by

$$g(z, w) = \begin{cases}\frac{f(z)-f(w)}{z-w} & \text{if } z \ne w \\ f'(z) & \text{if } z = w\end{cases}$$

Then g is continuous on $V \times V$. If we define $g_w(z) = g(z, w)$, then $g_w \in H(V)$ for all $w \in V$.

Proof. It is clear that g is continuous at (z, w) for $z \ne w$. Also, g_w is holomorphic at z for $z \ne w$. The fact that g is continuous at (z, z) follows directly from the holomorphicity of f. Now we need to show that g_z is holomorphic at z.

f is holomorphic, so it has power series expression at some disk $D(z, r)$.

$$f(w) = \sum_{n=0}^{\infty} c_n (w-z)^n \quad , \quad \text{for } w \in D(z, r)$$

$$\lim_{w\to z}\frac{g_z(w) - g_z(z)}{w - z} = \lim_{w\to z}\frac{\frac{f(w)-f(z)}{w-z} - f'(z)}{w - z} = \lim_{w\to z}\sum_{n=0}^{\infty} c_{n+2}(w-z)^n = c_2$$

So g_z is differentiable at z. ■

Theorem 7.4.4 (Cauchy Integral Formula, Homology Version)
Let $V \subseteq \mathbb{C}$ be open, $f \in H(V)$. Γ is a cycle in V with the property that

$$\operatorname{Ind}(\Gamma, a) = 0 \ , \quad \text{for every } a \in \mathbb{C} \setminus V \, .$$

Then for every $f \in H(V)$ and $z \in V \setminus \Gamma^*$,

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(w)\ dw}{w - z} = \operatorname{Ind}(\Gamma, z) f(z) \ .$$

Proof. We define g as Lemma 7.4.3. Now we define a new function h :

$$h(z) = \begin{cases} \int_\Gamma g(w,z)\ dw & \text{if } z \in V \\ \int_\Gamma \frac{f(w)\ dw}{w-z} & \text{if } z \notin \Gamma^* \text{ and } \operatorname{Ind}(\Gamma, z) = 0 \end{cases}$$

Since $\operatorname{Ind}(\Gamma, z) = 0$ for every $z \in C \setminus V$, h is defined on the whole $\mathbb{C}$. We need to show that h is well-defined. Suppose $z \in V$ and $z \notin \Gamma^*$ and $\operatorname{Ind}(\Gamma, z) = 0$. Then

$$\begin{aligned} \int_\Gamma g(w,z)\ dw &= \int_\Gamma \frac{f(w) - f(z)}{w - z}\ dw = \int_\Gamma \frac{f(w)\ dw}{w-z} - f(z) \int_\Gamma \frac{dw}{w-z} \\ &= \int_\Gamma \frac{f(w)\ dw}{w-z} - 2\pi i\, f(z) \operatorname{Ind}(\Gamma, z) = \int_\Gamma \frac{f(w)\ dw}{w-z} \end{aligned}$$

So h is well-defined.

$z \mapsto g(w,z)$ is holomorphic on V as proven in Lemma 7.4.3. Also, when $z \notin \Gamma^*$, $z \mapsto \frac{f(w)}{w-z}$ is holomorphic since $w \neq z$. Therefore, the pieces of h are holomorphic due to Corollary 6.3.9. Hence h is holomorphic everywhere. In other words, h is entire.

Γ^* is compact, so f is bounded on Γ^*. In other words, $|f| \leq M$ on Γ^*. Let l be the length of Γ. The unbounded component of $\mathbb{C} \setminus \Gamma^*$ is contained in the set where $\operatorname{Ind}(\Gamma, z) = 0$. So in this component, h is defined by the second expression. Consider z in this component.

$$|h(z)| = \left| \int_\Gamma \frac{f(w)\ dw}{w - z} \right| \leq \int_\Gamma \frac{|f(w)|\ |dw|}{|w-z|} \leq \frac{Ml}{d(\Gamma^*, z)}$$

As $z \to \infty$, $d(\Gamma^*, z) \to \infty$. So $h(z) \to 0$.

In other words, h is a bounded entire function. By Liouville's Theorem, h is constant. Since $h(z) \to 0$ for $z \to \infty$, h must an identically zero. Therefore, for $z \in V \setminus \Gamma^*$,

$$\begin{aligned} 0 = h(z) &= \int_\Gamma \frac{f(w) - f(z)}{w - z}\ dw = \int_\Gamma \frac{f(w)\ dw}{w-z} - f(z) \int_\Gamma \frac{dw}{w-z} \\ &= \int_\Gamma \frac{f(w)\ dw}{w-z} - 2\pi i\, f(z) \operatorname{Ind}(\Gamma, z) \end{aligned}$$

$$\therefore \frac{1}{2\pi i} \int_\Gamma \frac{f(w)\ dw}{w-z} = \operatorname{Ind}(\Gamma, z) f(z)$$

■

Theorem 7.4.5 (Cauchy's Theorem, Homology Version)
Let $V \subseteq \mathbb{C}$ be open, $f \in H(V)$. Γ is a cycle in V with the property that

$$\operatorname{Ind}(\Gamma, a) = 0\,, \quad \text{for every } a \in \mathbb{C} \setminus V\,.$$

Then for every $f \in H(V)$,

$$\int_\Gamma f(z)\ dz = 0\,.$$

Proof. We fix some $z \in V \setminus \Gamma^*$. Then we apply Homology version of Cauchy integral formula on the map $w \to (w - z) f(w)$ at $w = z$:

$$(z - z) f(z) \operatorname{Ind}(\Gamma, z) = \frac{1}{2\pi i} \int_\Gamma \frac{(w - z) f(w)\ dw}{w - z} \implies \int_\Gamma f(w)\ dw = 0$$

■

Definition 7.4.1 (Homologous Cycle). Suppose $V \subseteq \mathbb{C}$. Two cycles Γ_0 and Γ_1 in V are said to be **homologous** in V if $\operatorname{Ind}(\Gamma_0, z) = \operatorname{Ind}(\Gamma_1, z)$ for all $z \in \mathbb{C} \setminus V$.

This definition can be rephrased as

$$\operatorname{Ind}(\Gamma_0 - \Gamma_1, z) = 0\,, \quad \text{for all } z \in \mathbb{C} \setminus V$$

In particular, when we say that a cycle Γ is **homologous to zero** in V, we basically mean that

$$\operatorname{Ind}(\Gamma, z) = 0\,, \quad \text{for all } z \in \mathbb{C} \setminus V$$

Corollary 7.4.6
Let $V \subseteq \mathbb{C}$ be open and $f \in H(V)$. If two cycles Γ_0 and Γ_1 in V are homologous in V, then

$$\int_{\Gamma_0} f(z)\ dz = \int_{\Gamma_1} f(z)\ dz$$

Proof. Homology version of Cauchy's theorem on the cycle $\Gamma_0 - \Gamma_1$. ■

8 Singularity Points and Residue

We shall write $D'(z_0, r)$ to denote the punctured disk:

$$D'(z_0, r) = D(z_0, r) \setminus \{z_0\} = \{z \in \mathbb{C} \ : \ 0 < |z - z_0| < r\}$$

§8.1 Classification of Singularities

Definition 8.1.1 (Isolated Singularity)**.** The function $f \in H(V)$ has an **isolated singularity** at z if $D'(z, r) \subseteq V$ for some $r > 0$ but $z \notin V$.

Definition 8.1.2. Suppose f has an isolated singularity at z.

1. The singularity is **removable** if f can be defined at z such that f becomes holomorphic in a neighborhood of z.
2. The singularity is a **pole** if $\lim_{w \to z} f(w) = \infty$.
3. The singularity is **essential** if it is neither removable nor a pole.

Proposition 8.1.1

Let V be a domain, and $a \in V$. Suppose f is holomorphic on $V \setminus \{a\}$. If $\lim_{z \to a} (z - a) f(z) = 0$, then f has a removable singlarity at a.

Proof. We define $g : V \to \mathbb{C}$ as follows

$$g(z) = \begin{cases} (z-a)^2 f(z) & \text{if } z \neq a \\ 0 & \text{if } z = a \end{cases}$$

Then g is holomorphic on $V \setminus \{a\}$, because f is. We claim that g is holomorphic on V. For that we need to show that g is differentiable at a.

$$\lim_{z \to a} \frac{g(z) - g(a)}{z - a} = \lim_{z \to a} \frac{(z-a)^2 f(z)}{z - a} = \lim_{z \to a} (z - a) f(z) = 0$$

So $g'(a)$ exists, and it is equal to 0. Hence, g is holomorphic on a. So g has a power series about a on the ball $D(a, r)$ for some $r > 0$.

$$g(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \ , \quad \text{for } z \in D(a, r)$$

$g(a) = g'(a) = 0$ gives us $c_0 = c_1 = 0$. So for $z \in D'(a, r)$,

$$f(z) = \frac{g(z)}{(z-a)^2} = c_2 + c_3 (z - a) + c_4 (z - a)^2 + \cdots$$

So we can define $f(a) = c_2$. Thus f is analytic on $D(a, r)$. Analytic functions are holomorphic, so f is holomorphic on $D(a, r)$.

Therefore, we can define $f(a)$ in such a way that f is holomorphic on a neighborhood of a. So the singularity at a is removable. ■

Proposition 8.1.1 can be rephrased as: "If a holomorphic function is bounded near an isolated singularity, then the singularity is removable."

Proposition 8.1.2

Suppose $f \in H(D'(z, r))$. Then f has a pole at z if and only if there exists $N \in \mathbb{N}$ and complex numbers $(c_n)_{n \geq -N}$ such that $c_{-N} \neq 0$ and

$$f(w) = \sum_{n=-N}^{\infty} c_n (w - z)^n \ , \qquad \text{for } w \in D'(z, r)$$

Proof. The if direction is trivial, because

$$f(w) = \sum_{n=-N}^{\infty} c_n (w - z)^n \implies \lim_{w \to z} f(w) = \infty$$

For the converse, suppose f has a pole at z. Since f is unbounded near z, there exists $\rho \in (0, r)$ such that $|f| > 1$ in $D'(z, \rho)$.

If we take $g = \frac{1}{f}$, since f is nonzero on $D'(z, \rho)$, $g \in H(D'(z, \rho))$. Also $|g| < 1$, so g is bounded near its isolated singularity z. Therefore, by Proposition 8.1.1, the singularity of g at z is removable. Since f has a pole at z, $g(z) = 0$ makes g holomorphic on $D(z, \rho)$.

Certainly, g is not identically 0. So there exists $N \in \mathbb{N}$ and $h \in H(D(z, \rho))$ such that

$$g(w) = (w - z)^N h(w) \ , \quad \text{for } w \in D(z, \rho) \ \text{ and } h(z) \neq 0 \, .$$

g has no zeroes in $D'(z, \rho)$, so h has no zeroes in $D(z, \rho)$. As a result, $\frac{1}{h}$ is holomorphic on $D(z, \rho)$. Since holomorphic functions are analytic,

$$\frac{1}{h(w)} = \sum_{n=0}^{\infty} a_n (w - z)^n \ , \quad \text{for } w \in D(z, \rho)$$

Note that $a_0 = \frac{1}{h(z)} \neq 0$. As a result,

$$f(w) = \frac{1}{g(w)} = (w - z)^{-N} \frac{1}{h(w)} = (w - z)^{-N} \sum_{n=0}^{\infty} a_n (w - z)^n$$

for $w \in D'(z, \rho)$. This can be rewritten as

$$f(w) = \sum_{n=-N}^{\infty} c_n (w - z)^n$$

where $c_{-N} = a_0 \neq 0$. This series representation is true in $D'(z, \rho)$. $D'(z, \rho)$ has plenty of limit points in $D'(z, r)$. Therefore, by Corollary 6.3.5, the series representation is true in $D'(z, r)$. ■

Proposition 8.1.3

Suppose $f \in H(D'(z,r))$. Then f has an essential singularity at z if and only if $f(D'(z,\rho))$ is dense in $\mathbb{C}$ for all $\rho \in (0,r)$.

Proof. We shall prove the contrapositive statements in both directions.

($\Rightarrow$) If f does not have an essential singularity at z, then either z is a pole, or the singularity at z is removable. Either way $f(w)$ has a finite or infinite limit as $w \to z$. Hence, it's not true that $f(D'(z,\rho))$ is dense in $\mathbb{C}$ for all $\rho \in (0,r)$.

($\Leftarrow$) Now suppose that $f(D'(z,\rho))$ is not dense in $\mathbb{C}$ for some $\rho \in (0,r)$. We shall show that f does not have an essential singularity at z.

$f(D'(z,\rho))$ is not dense in $\mathbb{C}$, so there exists $\alpha \in \mathbb{C}$ which is not a limit point of $f(D'(z,\rho))$. Hence, we can find $\delta > 0$ such that

$$f(D'(z,\rho)) \cap D(\alpha,\delta) = \varnothing \implies |f(w) - \alpha| \geq \delta \quad \text{for } w \in D'(z,\rho)$$

In particular $f - \alpha$ has no zeroes in $D'(z,\rho)$. So $\frac{1}{f-\alpha} \in H(D'(z,\rho))$. We have

$$\left|\frac{1}{f(w) - \alpha}\right| \leq \frac{1}{\delta} \quad \text{for } w \in D'(z,\rho)$$

So $\frac{1}{f-\alpha}$ is bounded near its isolated singularity z. Therefore, by Proposition 8.1.1, the singularity of $\frac{1}{f-\alpha}$ at z is removable. In other words, there exists $g \in H(D(z,\rho))$ such that

$$g(w) = \frac{1}{f(w) - \alpha} \quad \text{for } w \in D'(z,\rho)$$

Hence $f = \alpha + \frac{1}{g}$ in $D'(z,\rho)$. If $g(z) \neq 0$, then $\alpha + \frac{1}{g} \in H(D(z,\rho))$. So f has a removable singularity at z.

Otherwise if $g(z) = 0$, then f approaches infinity at z. So f has a pole at z in this case. Either way, f does not have an essential singularity at z. ∎

To summarize, if f has an essential singularity at z_0, then for every $\alpha \in \mathbb{C}$, you can choose a sequence $(z_n)_{n\in\mathbb{N}}$ that converges to z_0, but

$$\lim_{n\to\infty} f(z_0) = \alpha$$

So f behaves really wildly near z_0.

When we have a removable singularity, then we can just assign a new value at that point and the function becomes holomorphic. When we have a pole, then we can just take $\frac{1}{f}$ and it is holomorphic at some neighborhood of thet pole. But if we have an essential singularity, we can't really do anything at all to make a holomorphic function. That's why it's called **"essential"** singularity.

§8.2 Laurent Series

We shall write $A(z_0,r,R)$ to denote the annulus:

$$A(z_0,r,R) = D(z_0,R) \setminus \overline{D(z_0,r)} = \{z \in \mathbb{C} \ : \ r < |z - z_0| < R\}$$

where $0 \leq r < R \leq \infty$. It turns out that if f is holomorphic on an annulus $A(z_0,r,R)$, then f has a *Laurent series* expansion:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \ , \qquad \text{for } z \in A(z_0,r,R)$$

Theorem 8.2.1

Suppose $a \in \mathbb{C}$ and $0 \leq r < R \leq \infty$. If $f \in H(A(a,r,R))$ then there exists a sequence of complex numbers $(c_n)_{n\in\mathbb{Z}}$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \ , \quad \text{for } z \in A(a,r,R) \ .$$

This series converges uniformly on compact subsets of $A(a,r,R)$. Also, the coefficients are given by

$$c_n = \frac{1}{2\pi i}\int_{\partial D(a,\rho)} \frac{f(w)\ dw}{(w-a)^{n+1}}$$

for any $\rho \in (r,R)$.

Proof. We choose r_1, r_2 such that $r < r_1 < r_2 < R$. Now we take the cycle

$$\Gamma = \partial D(a,r_2) - \partial D(a,r_1)$$

Now we claim that $\operatorname{Ind}(\gamma,z) = 0$ for $z \in \mathbb{C}\setminus A(a,r,R) = (\mathbb{C}\setminus D(a,R)) \cup \overline{D(a,r)}$.

If $z \in \overline{D(a,r)}$, then $\operatorname{Ind}(\partial D(a,r_2),z) = 1 = \operatorname{Ind}(\partial D(a,r_1),z)$. Hence $\operatorname{Ind}(\gamma,z) = 0$. Now if $z \in \mathbb{C}\setminus D(a,R)$, then $\operatorname{Ind}(\partial D(a,r_2),z) = 0 = \operatorname{Ind}(\partial D(a,r_1),z)$. Hence $\operatorname{Ind}(\gamma,z) = 0$. So our claim is proved.

Therefore, Γ is homologous to zero in $A(a,r,R)$. On the other hand, if $z \in A(a,r_1,r_2)$, then $\operatorname{Ind}(\partial D(a,r_2),z) = 1$ and $\operatorname{Ind}(\partial D(a,r_1),z) = 0$. So $\operatorname{Ind}(\Gamma,z) = 1$.

Let $\gamma_1 = \partial D(a,r_1)$ and $\gamma_2 = \partial D(a,r_2)$, By Homology version of Cauchy integral formula, for $z \in A(a,r_1,r_2)$,

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(w)\ dw}{w-z} = \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(w)\ dw}{w-z} - \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)\ dw}{w-z}$$

Now let's expand the integrals separately. If $w \in \gamma_2^*$, $\left|\frac{z-a}{w-a}\right| = \frac{|z-a|}{r_2} < 1$.

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(w)\ dw}{w-z} &= \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(w)}{w-a}\frac{1}{1-\frac{z-a}{w-a}}\,dw \\
&= \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(w)}{w-a}\left(\sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n\right) dw \\
&= \sum_{n=0}^{\infty}(z-a)^n\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(w)\ dw}{(w-a)^{n+1}} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(w)\ dw}{(w-a)^{n+1}}\right)(z-a)^n
\end{aligned}$$

Here swapping integral and summation is justified because of UCT. The argument is similar to that of power series.

In a similar manner, if $w \in \gamma_1$, $\left|\frac{w-a}{z-a}\right| = \frac{r_1}{|z-a|} < 1$. Hence,

$$\begin{aligned}
-\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)\ dw}{w-z} &= \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{z-a}\frac{1}{1-\frac{w-a}{z-a}}\ dw \\
&= \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{z-a}\left(\sum_{m=0}^{\infty}\left(\frac{w-a}{z-a}\right)^m\right)\ dw \\
&= \sum_{m=0}^{\infty}(z-a)^{-m-1}\frac{1}{2\pi i}\int_{\gamma_1}f(w)(w-a)^m\ dw \\
&= \sum_{n=-\infty}^{-1}(z-a)^n\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)\ dw}{(w-a)^{n+1}} \\
&= \sum_{n=-\infty}^{-1}\left(\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)\ dw}{(w-a)^{n+1}}\right)(z-a)^n
\end{aligned}$$

Here, too, swapping integral and summation is justified because of UCT.

Given any $\rho \in (r, R)$, the cycle $\partial D(a,\rho) - \partial D(a, r_1)$ is homologous to zero in $A(a, r, R)$. Also, the map $w \mapsto \frac{f(w)}{(w-a)^{n+1}}$ is holomorphic on $A(a, r, R)$. Therefore, by Homology version of Cauchy's theorem,

$$\int_{\partial D(a,\rho)-\partial D(a,r_1)}\frac{f(w)}{(w-a)^{n+1}}\ dz = 0 \implies \int_{\partial D(a,\rho)}\frac{f(w)}{(w-a)^{n+1}}\ dw = \int_{\gamma_1}\frac{f(w)}{(w-a)^{n+1}}\ dw$$

Similarly, for γ_2, we have

$$\int_{\partial D(a,\rho)}\frac{f(w)}{(w-a)^{n+1}}\ dw = \int_{\gamma_2}\frac{f(w)}{(w-a)^{n+1}}\ dw$$

Therefore, combining the two sums, we have

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial D(a,\rho)}\frac{f(w)\ dw}{(w-a)^{n+1}}\right)(z-a)^n + \sum_{n=-\infty}^{1}\left(\frac{1}{2\pi i}\int_{\partial D(a,\rho)}\frac{f(w)\ dw}{(w-a)^{n+1}}\right)(z-a)^n$$

Now, about convergence, the geometric series used in the first sum converges uniformly and absolutely when $\left|\frac{z-a}{w-a}\right| = \frac{|z-a|}{r_2} < 1$. In other words, the first sum converges uniformly absolutely in compact subsets of $D(a, r_2)$.

Similarly, the geometric series used in the second sum converges uniformly and absolutely when $\left|\frac{w-a}{z-a}\right| = \frac{r_1}{|z-a|} < 1$. In other words, the second sum converges uniformly absolutely in compact subsets of $\mathbb{C} \setminus \overline{D(a, r_1)}$.

Therefore, the Laurent series expansion converges uniformly absolutely in compact subsets of $D(a, r_2) \cap \mathbb{C} \setminus \overline{D(a, r_1)} = A(a, r_1, r_2)$. r_1 and r_2 were arbitrary satisfying $r < r_1 < r_2 < R$. So the Laurent series expansion converges uniformly absolutely in compact subsets of $A(a, r, R)$.

Now we are left with the uniqueness part. Suppose there are another sequence $(d_n)_{n\in\mathbb{Z}}$ that satisfies

$$f(z) = \sum_{n=-\infty}^{\infty} d_n (z-a)^n \ , \quad \text{for } z \in A(a, r, R) \ ,$$

converging absolutely uniformly on the compact subsets of $A(a, r, R)$. Then

$$c_n = \frac{1}{2\pi i}\int_{\partial D(a,\rho)}\sum_{m=-\infty}^{\infty} d_m (w-a)^m \frac{dw}{(w-a)^{n+1}}$$

Since the sum converges uniformly, we can swap summation and integral.

$$c_n = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} d_m \int_{\partial D(a,\rho)} (w-a)^{m-n-1}\, dw$$

When $m \neq n$, each terms of the form $(w-a)^{m-n-1}$ have primitives. So they don't conbtribute anything to the integral. Therefore,

$$c_n = \frac{1}{2\pi i} d_n \int_{\partial D(a,\rho)} \frac{dw}{w-a} = d_n \operatorname{Ind}\left(\partial D\left(a,\rho\right), a\right) = d_n$$

So, the coefficients are unique. ∎

Note that $D'(z_0, r) = A(z_0, 0, r)$. So if a function is holomorphic on the punctured disk $D'(z_0, r)$, then it has a Laurent series expansion.

Proposition 8.2.2

Suppose $f \in H\left(D'\left(z_0, r\right)\right)$, so it has the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n \left(z - z_0\right)^n \ , \qquad \text{for } z \in D'\left(z_0, r\right)$$

(i) f has a removable singularity at z_0 if and only if $c_n = 0$ for all $n < 0$.

(ii) f has a pole of order $N > 0$ at z_0 if and only if $c_{-N} \neq 0$ and $c_n = 0$ for every $n < -N$.

(iii) f has an essential singularity at z_0 if and only if there exists infinitely many values of $n < 0$ such that $c_n \neq 0$.

Proof. (i) If f has a removable singularity at z_0, then we can define $f(z_0)$ such that $f \in H\left(D\left(z_0, r\right)\right)$. Then f has a power series representation about z_0. The coefficients of power series agrees with the coefficients of Laurent series (Theorem 8.2.1). Hence, for $n < 0$, c_n cannot be nonzero.

For the reverse direction, suppose all the c_n are zero for $n < 0$. Then if we define $f(z_0) = c_0$, then f is analytic on $D(z_0, r)$. Hence, f is holomorphic on $D(z_0, r)$, and the singularity at z_0 is removable.

(ii) This is just a restatement of Proposition 8.1.2.

(iii) If $n < 0$, we shall call c_n a "negative index coefficient". If f has an essential singularity at z_0, then z_0 is neither a removable singularity nor a pole. Thus not all the negative index coefficients can be 0. Furthermore, it cannot happen that finitely many negative index coefficients are nonzero (because that would mean that z_0 is a pole). So it must be the case of infinitely many negative index coefficients are nonzero.

For the reverse direction, suppose infinitely many negative index coefficients are nonzero. Then z_0 cannot be a removable singularity because not all the negative index coefficients are 0. Also z_0 cannot be a pole since infinitely many negative index coefficients are nonzero. So z_0 is an essential singularity.

∎

§8.3 Residue

By this time , you're probably convinced that holomorphic functions are the nicest possible function. They have lots of cool properties. We have the second nicest kind of function, namely **meromorphic function**.

Definition 8.3.1 (Meromorphic Function)**.** A function f on an open set V is said to be **meromorphic** if there exists a sequence of points $\{z_1, z_2, \ldots\}$ that has no limit points in V, and

i. f is holomorphic in $V \setminus \{z_1, z_2, \ldots\}$.

ii. f has poles at the points $\{z_1, z_2, \ldots\}$.

It's of no surprise that Cauchy's theorem does not hold for meromorphic functions. The motivation between defining "residue" is to measure how much the meromorphic function deviate from satisfying Cauchy's theorem.

Definition 8.3.2 (Residue)**.** Suppose f has an isolated singularity at z_0, and it has the Laurent series expansion in $D'(z_0, r)$ for some $r > 0$.

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \quad , \quad \text{for } z \in D'(z_0, r) \; .$$

Then the **residue** of f at z_0 is defined as

$$\operatorname{Res}(f, z_0) = c_{-1} = \text{the coefficient of } \frac{1}{z - z_0}$$

Now, why does this thing measure how much Cauchy's theorem fails? If we integrate f over some closed curve γ, then all the terms $(z - z_0)^n$ (where $n \geq 0$) have primitives. So they contribute 0 to the integral. Also, all the terms $\frac{1}{(z-z_0)^n}$ for $n \geq 2$ have primitives. So they also contribute 0 to the integral. The only thing which has issues with having primitive is the term $\frac{1}{z-z_0}$. So, you should expect to get

$$\int_\gamma f(z) \; dz = c_{-1} \, .$$

That's why this coefficient is of importance.

If f has a removable singularity, then obviously $\operatorname{Res}(f, z_0) = 0$ (Proposition 8.2.2). If f has a pole of order N at z_0, we have

$$f(z) = \sum_{n=-N}^{\infty} c_n (z - z_0)^n$$

near z_0. Let $g(z) = (z - z_0)^N f(z)$. Then g has a removable singularity at z_0, since

$$g(z) = (z - z_0)^N f(z) = (z - z_0)^N \sum_{n=-N}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} c_{n-N} (z - z_0)^n$$

Then $\operatorname{Res}(f, z_0) = c_{-1}$ is equal to the coefficient of $(z - z_0)^{N-1}$ in the power series of g about z_0. As a result,

$$\operatorname{Res}(f, z_0) = c_{-1} = \frac{g^{(N-1)}(z_0)}{(N-1)!}$$

In particular, if f has a simple pole (pole of order 1) at z_0, then the residue of f at z_0 is equal to $g(z_0)$.

$$\operatorname{Res}(f, z_0) = g(z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

We've seen before that if f has a removable singularity at z_0, $\operatorname{Res}(f, z_0)$ is 0. By Proposition 8.1.1,

$$\operatorname{Res}(f, z_0) = 0 = \lim_{z \to z_0} (z - z_0) f(z)$$

Thus we have the following result:

Lemma 8.3.1

If f has either a removable singularity or a simple pole at z_0, then

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

Now let's calculate some residues.

Example 8.3.1. Suppose f, g are holomorphic near z_0 and f has a simple zero (zero of order 1) at z_0. We want to calculate $\operatorname{Res}(g/f, z_0)$.

If $g(z_0) = 0$, then g/f has a removable singularity at z_0. Otherwise, g/f has a simple pole at z_0. Either way, we can apply Lemma 8.3.1.

$$\begin{aligned}\operatorname{Res}(g/f, z_0) &= \lim_{z \to z_0} (z - z_0) \frac{g(z)}{f(z)} = \lim_{z \to z_0} (z - z_0) \frac{g(z)}{f(z) - f(z_0)} \\ &= \left(\lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} \right) \left(\lim_{z \to z_0} g(z) \right) = \frac{g(z_0)}{f'(z_0)}\end{aligned}$$

Note that, $f'(z_0)$ is nonzero. Because if $f'(z_0)$ were 0, it would mean that f is identically 0 in a neighborhood of z_0. This contradicts with the fact that f has a simple zero at z_0.

Example 8.3.2. Suppose g is holomorphic near z_0 and f has a simple pole at z_0. We want to find $\operatorname{Res}(fg, z_0)$.

If $g(z_0) = 0$, then fg has a removable singularity at z_0. So $\operatorname{Res}(fg, z_0) = 0$.

But if $g(z_0) \neq 0$, then fg has a simple pole at z_0. Then using Lemma 8.3.1, we get that

$$\begin{aligned}\operatorname{Res}(fg, z_0) &= \lim_{z \to z_0} (z - z_0) f(z) g(z) = \left(\lim_{z \to z_0} (z - z_0) f(z) \right) \left(\lim_{z \to z_0} g(z) \right) \\ &= \operatorname{Res}(f, z_0) g(z_0)\end{aligned}$$

Therefore, in all the cases $\operatorname{Res}(fg, z_0) = \operatorname{Res}(f, z_0) g(z_0)$.

Example 8.3.3. Suppose f has an isolated singularity at z_0. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be a curve given by $\gamma(t) = z_0 + re^{int}$ for some fixed $n \in \mathbb{Z}$. We want to compute the following integral:

$$\int_\gamma f(z) \, dz$$

Since f has an isolated singularity at z_0, it has a Laurent series expression about z_0.

$$\int_\gamma f(z)\ dz = \int_\gamma \sum_{n=\infty}^{\infty} c_n (z-z_0)^n\ dz = \sum_{n=\infty}^{\infty} \int_\gamma c_n (z-z_0)^n\ dz$$

Here we swapped integral and summation. Since Laurent series converges uniformly, UCT allows this swap.

$$\int_\gamma f(z)\ dz = \sum_{n=2}^{\infty} \int_\gamma \frac{c_{-n}\,dz}{(z-z_0)^n} + \int_\gamma \frac{c_{-1}\,dz}{z-z_0} + \sum_{n=0}^{\infty} \int_\gamma c_n (z-z_0)^n\ dz$$

Now, each of the terms $\frac{1}{(z-z_0)^n}$ has a primitive for $n \geq 2$. So the integral of them over γ is 0. Also, each terms of the form $(z-z_0)^n$ has a primitive for $n \geq 0$. So, they also contribute 0 to the integral. Therefore,

$$\int_\gamma f(z)\ dz = c_{-1} \int_\gamma \frac{dz}{z-z_0} = \operatorname{Res}(f, z_0)\, 2\pi i \operatorname{Ind}(\gamma, z_0)$$

$$\therefore \frac{1}{2\pi i} \int_\gamma f(z)\ dz = \operatorname{Ind}(\gamma, z_0) \operatorname{Res}(f, z_0)$$

This is a special case of a famous result called *Residue theorem*.

§8.4 Residue Theorem and Residue Calculus

Definition 8.4.1 (Principal Part)**.** Suppose f is holomorphic on $D'(a, r)$. Then it has a Laurent series expression:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \ , \quad \text{for } z \in D'(a, r)$$

Then we define

$$P_a(f) = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n}$$

to be the **principal part** of f at a.

Theorem 8.4.1 (Residue Theorem)

Let $V \subseteq \mathbb{C}$ be open, and S be a finite subset of V. Suppose a cycle Γ in V is homologous to zero. Suppose $f \in H(V \setminus S)$. Then

$$\frac{1}{2\pi i} \int_\Gamma f(z)\ dz = \sum_{p \in S} \operatorname{Ind}(\Gamma, p) \operatorname{Res}(f, p)$$

Proof. Let $w_1, w_2, \ldots, w_n$ be the elements of S. We take positive numbers $r_1, r_2, \ldots, r_n$ such that the closed disks $\overline{D(w_j, r_j)}$ are all pairwise disjoint and $\overline{D(w_j, r_j)} \subseteq V$ for all $j = 1, 2, \ldots, n$. Let $\gamma_j = \partial D(w_j, r_j)$, and $m_j = \operatorname{Ind}(\Gamma, w_j)$. We define a new cycle Λ as follows:

$$\Lambda = \Gamma - m_1\gamma_1 - m_2\gamma_2 \cdots - m_n\gamma_n$$

We claim that $\operatorname{Ind}(\Lambda, z) = 0$ for all $z \in \mathbb{C} \setminus (V \setminus S) = (\mathbb{C} \setminus V) \cup S$.

$$\operatorname{Ind}(\Lambda, z) = \operatorname{Ind}(\Gamma, z) - m_1 \operatorname{Ind}(\gamma_1, z) - m_2 \operatorname{Ind}(\gamma_2, z) \cdots - m_n \operatorname{Ind}(\gamma_n, z)$$

If $z \in \mathbb{C} \setminus V$, then $\operatorname{Ind}(\Gamma, z) = 0$ since Γ is homologous to zero in V. $\overline{D(w_j, r_j)} \subseteq V$, so $\operatorname{Ind}(\gamma_j, z) = \operatorname{Ind}(\partial D(w_j, r_j), z) = 0$. So $\operatorname{Ind}(\Lambda, z) = 0$.

If $z \in S$, $z = w_k$ for some k. Then $\operatorname{Ind}(\gamma_j, w_k) = 0$ if $j \neq k$, because $D(w_j, r_j)$ does not contain w_k. Also $\operatorname{Ind}(\gamma_k, w_k) = 0$.

$$\operatorname{Ind}(\Lambda, w_k) = \operatorname{Ind}(\Gamma, w_k) - m_k \operatorname{Ind}(\gamma_k, w_k) = \operatorname{Ind}(\Gamma, w_k) - \operatorname{Ind}(\Gamma, w_k) = 0$$

So our claim holds. Therefore, Λ is homologous to zero in $V \setminus S$. f is holomorphic in $V \setminus S$, so applying Homology version of Cauchy's theorem, we get

$$\begin{aligned}
\int_\Lambda f(z)\ dz = 0 &\implies \frac{1}{2\pi i}\int_\Lambda f(z)\ dz = 0 \\
&\implies \frac{1}{2\pi i}\int_\Gamma f(z)\ dz - \frac{1}{2\pi i}\sum_{j=1}^{n} m_j \int_{\gamma_j} f(z)\ dz \\
&\implies \frac{1}{2\pi i}\int_\Gamma f(z)\ dz = \sum_{j=1}^{n} \operatorname{Ind}(\Gamma, w_j)\, \frac{1}{2\pi i}\int_{\gamma_j} f(z)\ dz
\end{aligned}$$

Using Example 8.3.3,

$$\frac{1}{2\pi i}\int_{\gamma_j} f(z)\ dz = \operatorname{Ind}(\gamma_j, w_j) \operatorname{Res}(f, w_j) = \operatorname{Res}(f, w_j)$$

Therefore, we get

$$\frac{1}{2\pi i}\int_\Gamma f(z)\ dz = \sum_{j=1}^{n} \operatorname{Ind}(\Gamma, w_j) \operatorname{Res}(f, w_j) = \sum_{p \in S} \operatorname{Ind}(\Gamma, p) \operatorname{Res}(f, p)$$

■

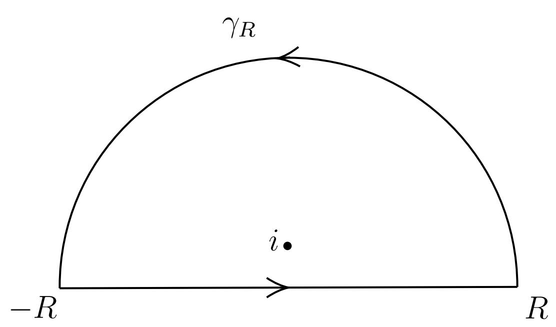
Residue theorem is a powerful tool to compute integrals.

Example 8.4.1. We shall compute the following integral:

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$$

This is a well-known integral, so it's probably not a great example to illustrate residue theorem's power. But we will see a "better" example soon.

Consider $f(z) = \frac{1}{z^2}$, we shall integrate f over the following contour $\Gamma = \gamma_R + [-R, R]$, where γ_R is the semicircular arc.



Now, $f(z) = \frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$. f has a simple pole at i and $-i$. The region enclosed by Γ contains one singularity point i. So we need to find $\operatorname{Res}(f, i)$.

$$\operatorname{Res}(f, i) = \lim_{z \to i} (z - i) f(z) = \lim_{z \to i} \frac{1}{z + i} = \frac{1}{2i}$$

Also $\operatorname{Ind}(\Gamma, i) = 1$. Therefore, by Residue Theorem,

$$\int_{\Gamma} f(z)\ dz = 2\pi i \operatorname{Ind}(\Gamma, i) \operatorname{Res}(f, i) = \pi \implies \int_{-R}^{R} \frac{dx}{1 + x^2} + \int_{\gamma_R} f(z)\ dz = \pi$$

Taking $R \to \infty$, we get

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} + \lim_{R \to \infty} \int_{\gamma_R} f(z)\ dz = \pi$$

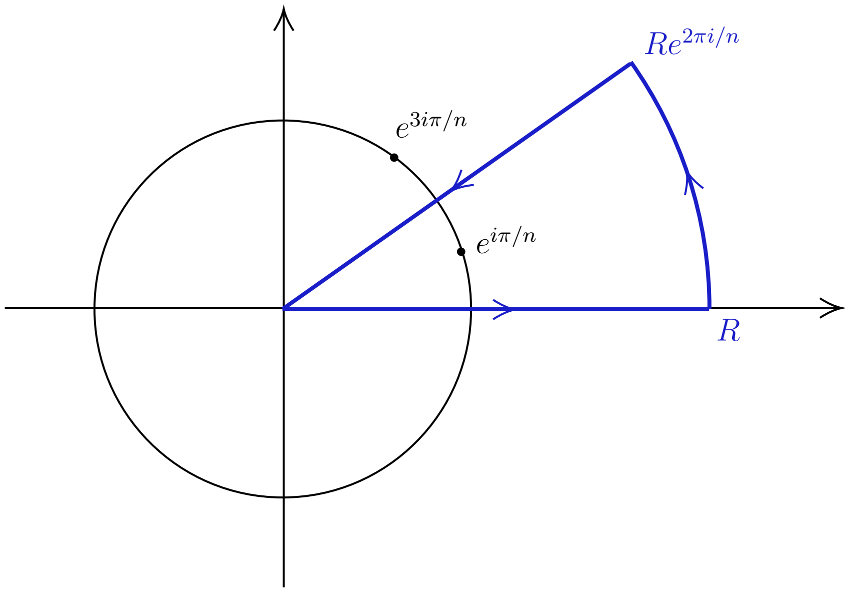
Now we shall show that $\int_{\gamma_R} f$ goes to 0 as $R \to \infty$. $|f(z)| \le \frac{B}{|z|^2}$ for some suitable B. So f is bounded by $\frac{B}{R^2}$ on γ_R. Also, the length of γ_R is πR. So by ML Inequality,

$$\left| \int_{\gamma_R} f(z)\ dz \right| \le \frac{B}{R^2} \pi R = \frac{B\pi}{R} \implies \lim_{R \to \infty} \int_{\gamma_R} f(z)\ dz = 0 \implies \boxed{\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \pi}$$

Example 8.4.2. Now we want to generalize Example 8.4.1. We shall compute the following integral

$$\int_{0}^{\infty} \frac{dx}{1 + x^2}\ , \quad \text{where } n \ge 2$$

It's not a surprise that we would consider the function $f(z) = \frac{1}{1+z^n}$. $z^n + 1$ is 0 when $z = e^{i\pi \frac{2m+1}{n}}$ where $m = 0, 1, \ldots, n-1$. So f has a simple pole at each $e^{i\pi \frac{2m+1}{n}}$. Now we need to choose a suitable contour. We shall integrate f over the following "sector of circle" contour (colored in blue).



We name this contour Γ_R. So $\Gamma_R = [0, R] + \gamma_R + \left[Re^{2\pi i/n}, 0\right]$, where γ_R is the circular arc. Let $\zeta = e^{i\pi/n}$. Then ζ is a singularity of f. $\operatorname{Ind}(\Gamma_R, \zeta) = 1$. So by Residue Theorem,

$$\int_{\Gamma_R} f(z)\ dz = \int_{0}^{R} \frac{dx}{1 + x^n} + \int_{\gamma_R} f(z)\ dz - \int_{\left[0, Re^{2\pi i/n}\right]} f(z)\ dz = 2\pi i \operatorname{Res}(f, \zeta)$$

Let's compute the third integral now. $\left[0, Re^{2\pi i/n}\right]$ can be paramtrized by $\gamma(t) = te^{2\pi i/n}$ for $0 \leq t \leq R$.

$$\int_{\left[0,Re^{2\pi i/n}\right]} f(z)\ dz = \int_0^R \frac{e^{2\pi i/n}\ dt}{1+\left(te^{2\pi i/n}\right)^n} = e^{2\pi i/n}\int_0^R \frac{dt}{1+t^n}$$

On γ_R, $|z^n+1| \geq |z^n| - 1 = R^n - 1$. The length of γ_R is smaller than $2\pi R$. Therefore, by ML Inequality,

$$\left|\int_{\gamma_R} f(z)\ dz\right| \leq \frac{2\pi R}{R^n+1} \implies \lim_{R\to\infty}\int_{\gamma_R} f(z)\ dz = 0$$

If $R \to \infty$, $\operatorname{Res}(f,\zeta)$ does not change. Therefore,

$$2\pi i \operatorname{Res}(f,\zeta) = \lim_{R\to\infty} f(z)\ dz = \int_0^\infty \frac{dx}{1+x^n} - e^{2\pi i/n}\int_0^\infty \frac{dt}{1+t^n} = \left(1-e^{2\pi i/n}\right)\int_0^\infty \frac{dx}{1+x^n}$$

Now we need to compute $\operatorname{Res}(f,\zeta)$. $1+z^n$ has a simple zero at ζ. Therefore, by Example 8.3.1,

$$\operatorname{Res}(f,\zeta) = \frac{1}{n\zeta^{n-1}} = -\frac{\zeta}{n}$$

$e^{2\pi i/n} = \zeta^2$, so we have

$$\int_0^\infty \frac{dx}{1+x^n} = -\frac{1}{1-\zeta^2}2\pi i\frac{\zeta}{n} = \frac{\pi}{n}\frac{2i}{z-z^{-1}}$$

After simplifying things, we get

$$\frac{z-z^{-1}}{2i} = \frac{e^{i\pi/n}-e^{-i\pi/n}}{2i} = \sin\left(\frac{\pi}{n}\right) \implies \boxed{\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi/n}{\sin(\pi/n)}}$$

Lemma 8.4.2

Suppose f is holomorphic on $D'(z_0,r)$ and f has a simple pole at z_0. For $\epsilon > 0$, let $C_\epsilon : [0,\pi] \to \mathbb{C}$ be defined by $C_\epsilon(t) = z_0 + \epsilon e^{it}$. Then

$$\lim_{\epsilon\to 0^+}\int_{C_\epsilon} f(z)\ dz = i\pi \operatorname{Res}(f,z_0)$$

Proof. The pole on z_0 has order 1, so there exists $g : D(z_0,r) \to \mathbb{C}$ such that

$$f(z) = \frac{g(z)}{z-z_0}\ ,\ \text{and } g \text{ is holomorphic on } D(z_0,r).$$

Since f has a simple pole on z_0,

$$\operatorname{Res}(f,z_0) = \lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to z_0} g(z) = g(z_0)$$

$$\int_{C_\epsilon} f(z)\ dz = \int_{C_\epsilon}\frac{g(z)\ dz}{z-z_0} = \int_0^\pi \frac{g\left(z_0+\epsilon e^{it}\right)}{\epsilon e^{it}} i\epsilon e^{it}\,dt = i\int_0^\pi g\left(z_0+\epsilon e^{it}\right)\ dt$$

$$\begin{aligned}\left|\int_{C_\epsilon} f(z)\ dz - i\pi \operatorname{Res}(f,z_0)\right| &= \left| i\int_0^\pi g\left(z_0+\epsilon e^{it}\right)\ dt - i\pi g(z_0)\right| \\ &= \left|\int_0^\pi g\left(z_0+\epsilon e^{it}\right)\ dt - \int_0^\pi g(z_0)\ dt\right| \\ &= \left|\int_0^\pi \left(g\left(z_0+\epsilon e^{it}\right) - g(z_0)\right)\ dt\right| \\ &\le \int_0^\pi \left|g\left(z_0+\epsilon e^{it}\right) - g(z_0)\right|\ dt\end{aligned}$$

As $\epsilon \to 0^+$, $g\left(z_0+\epsilon e^{it}\right) \to g(z_0)$. Also, C_ϵ is compact, so $g\left(z_0+\epsilon e^{it}\right)$ is bounded. Hence, $\left|g\left(z_0+\epsilon e^{it}\right) - g(z_0)\right|$ is bounded. Hence, switching limit and integral is valid. Therefore,

$$\begin{aligned}\lim_{\epsilon\to 0^+}\left|\int_{C_\epsilon} f(z)\ dz - i\pi \operatorname{Res}(f,z_0)\right| &\le \lim_{\epsilon\to 0^+}\int_0^\pi \left|g\left(z_0+\epsilon e^{it}\right) - g(z_0)\right|\ dt \\ &= \int_0^\pi \lim_{\epsilon\to 0^+}\left|g\left(z_0+\epsilon e^{it}\right) - g(z_0)\right|\ dt = 0\end{aligned}$$

■

Example 8.4.3. Now we shall compute the following integral:

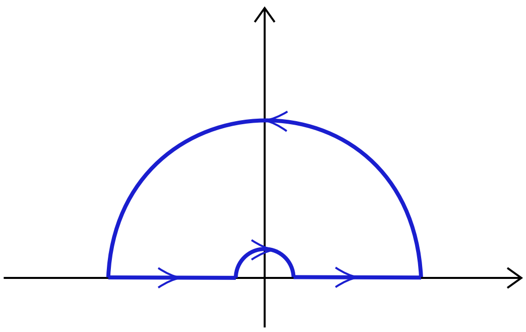
$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\ dx$$

Consider the function $f(z) = \frac{e_{iz}}{z}$. Then f has a pole at $z = 0$. The order of this pole is exactly 1, because $(z-0) f(z) = e^{iz}$ is entire.

$$f(z) = \frac{g(z)}{z-0}\ , \quad \text{where } g(z) = e^{iz} \text{ is entire}$$

so 0 is a simple pole.

Consider the following contour γ:



where the bigger semi-circular arc has radius M and the smaller semicircular arc has radius ε. In other words,

$$\gamma = C_M + [-M, -\varepsilon] - C_\varepsilon + [\varepsilon, M]$$

We claim that $\lim_{M\to\infty} \int_{C_M} f = 0$.

Using the definition of integral,

$$\int_{C_M} f(z)\ dz = \int_0^\pi \frac{e^{iM\cos t - M\sin t}}{Me^{it}}\ iMe^{it}\, dt = i\int_0^\pi e^{iM\cos t - M\sin t}\ dt$$

$$\therefore \left|\int_{C_M} f(z)\ dz\right| \le \int_0^\pi \left|e^{iM\cos t - M\sin t}\right|\ dt = \int_0^\pi e^{-M\sin t}\ dt = 2\int_0^{\frac{\pi}{2}} e^{-M\sin t}\ dt$$

$\sin'' = -\sin$, so sin is concave on $\left[0, \frac{\pi}{2}\right]$, so we have

$$\sin t = \sin\left(\frac{2t}{\pi}\frac{\pi}{2} + \left(1 - \frac{2t}{\pi}\right)0\right) \geq \frac{2t}{\pi}\sin\frac{\pi}{2} + \left(1 - \frac{2t}{\pi}\right)\sin 0 = \frac{2t}{\pi}$$

Therefore, $-M\sin t \leq -\frac{2Mt}{\pi}$, so $e^{-M\sin t} \leq e^{-\frac{2Mt}{\pi}}$.

$$\left|\int_{C_M} f(z)\ dz\right| \leq 2\int_0^{\frac{\pi}{2}} e^{-M\sin t}\ dt \leq 2\int_0^{\frac{\pi}{2}} e^{-\frac{2Mt}{\pi}}\ dt = 2\left[e^{-\frac{2Mt}{\pi}}\frac{-\pi}{2M}\right]_0^{\frac{\pi}{2}}$$
$$= -\frac{\pi}{M}e^{-M} + \frac{\pi}{M} = \frac{\pi}{M}\left(1 - e^{-M}\right) \leq \frac{\pi}{M}$$
$$\therefore \lim_{M\to\infty}\left|\int_{C_M} f(z)\ dz\right| \leq \lim_{M\to\infty}\frac{\pi}{M} = 0$$

So our claim is proved.

Using Lemma 8.4.2, we get

$$\lim_{\varepsilon\to 0^+}\int_{C_\varepsilon} f(z)\ dz = i\pi\operatorname{Res}(f, 0) = i\pi\lim_{z\to 0} zf(z) = i\pi\lim_{z\to 0} e^{iz} = i\pi$$

f is holomorphic on γ^* and on the region enclosed by γ. Therefore, we can find a star convex open set containing γ^* and the region enclosed by γ, such that f is holomorphic on that open set. Therefore, $\int_\gamma f = 0$. As a result $\int_\gamma f(z)\ dz$ is 0 as $M \to \infty$ and $\varepsilon \to 0^+$.

$$0 = \lim_{M\to\infty,\varepsilon\to 0^+}\left(\int_{C_M} f(z)\ dz + \int_{-M}^{-\varepsilon} f(x)\ dx - \int_{C_\varepsilon} f(z)\ dz + \int_\varepsilon^M f(x)\ dx\right)$$
$$= \lim_{M\to\infty}\int_{C_M} f(z)\ dz - \lim_{\varepsilon\to 0^+}\int_{C_\varepsilon} f(z)\ dz + \int_{-\infty}^{0^-} f(x)\ dx + \int_{0^+}^{\infty} f(x)\ dx$$
$$= 0 - i\pi + \int_{-\infty}^{\infty}\frac{\cos x + i\sin x}{x}\ dx \implies \int_{-\infty}^{\infty}\frac{\cos x\, dx}{x} + i\int_{-\infty}^{\infty}\frac{\sin x\, dx}{x} = i\pi$$

Equating the imaginary parts, we get

$$\boxed{\int_{-\infty}^{\infty}\frac{\sin x}{x}\ dx = \pi}$$

9 Open Mapping Theorem

In Problem 2(b) of HW2, we proved that f is constant on a domain $V \subseteq C$ if either of these are satisfied:

(i) $\operatorname{Re} f$ is constant.

(ii) $\operatorname{Im} f$ is constant.

(iii) $|f|$ is constant.

(iv) $\arg f$ is constant.

In this chapter we shall see a theorem, namely the "open mapping theorem", that trivializes the problem.

§9.1 Counting Zeros

If f is a holomorphic function, the zero set of f is denoted by Z_f. In this section, we shall consider Z_f to be a "multiset" rather than a set. This means that if an element occurs multiple times in Z_f, we shall count it multiple times.

Theorem 9.1.1

Suppose $f \in H(V)$. Let γ be a curve in V such that $\operatorname{Ind}(\gamma, z)$ is either 0 or 1 for $z \in \mathbb{C} \setminus \gamma^*$; and $\operatorname{Ind}(\gamma, z) = 0$ for $z \in C \setminus V$. Suppose f has no zeroes on γ^*, and let $\Omega = \{z \in V \ : \ \operatorname{Ind}(\gamma, z) = 1\}$. Then

$$\#(Z_f \cap \Omega) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)\ dz}{f(z)}$$

Proof. We define $g \in H(V \setminus Z_f)$ by

$$g(z) = \frac{f'(z)}{f(z)}$$

Z_f is closed in V with no limit points. Therefore, by Residue Theorem,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)\ dz}{f(z)} = \frac{1}{2\pi i} \int_\gamma g(z)\ dz = \sum_{z \in \Omega \cap Z_f} \operatorname{Ind}(\gamma, z) \operatorname{Res}(g, s) = \sum_{z \in \Omega \cap Z_f} \operatorname{Res}(g, s)$$

Suppose $z \in Z_f$ is a zero of f with order n. Since Z_f is isolated, there exists a small disk $D(z, r)$ such that f is nonzero on $D'(z, r)$. So g is holomorphic on $D'(z, r)$.

Also, the zero at z has order n. Hence, there exists $h \in H(V)$ such that $f(w) = (w - z)^n h(w)$, where $h(z) \neq 0$. f is nonzero on $D'(z, r)$, so h is nonzero on $D(z, r)$. Therefore, h'/h is holomorphic on $D(z, r)$. For $w \in D'(z, r)$,

$$g(w) = \frac{f'(w)}{f(w)} = \frac{n}{w - z} + \frac{h'(w)}{h(w)}$$

Therefore, $\text{Res}(g, z) = n$. Each zero in Ω contributes its order to the sum $\sum_{z\in\Omega\cap Z_f} \text{Res}(g, s)$. Hence, the number of zeroes (counting multiplicity) in Ω is

$$\sum_{z\in\Omega\cap Z_f} \text{Res}(g, s) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)\ dz}{f(z)}$$

■

However, we can extend this result for meromorphic functions. If f has a pole of order n at z, then $f(w) = (w-z)^{-n} h(w)$ near z, where h is holomorphic with $h(z) \neq 0$. Then carrying out the same computations, we can prove that

$$\text{number of zeroes} - \text{number of poles} = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)\ dz}{f(z)}$$

Corollary 9.1.2 (Argument Principle)

Suppose f, γ, etc. are as in Theorem 9.1.1. We define a curve $\widetilde{\gamma}$ by $\widetilde{\gamma}(t) = f(\gamma(t))$. Then the number of zeroes in Ω is $\text{Ind}(\widetilde{\gamma}, 0)$.

Proof. $\gamma : [a, b] \to \mathbb{C}$. Then by the definition of Ind,

$$\begin{aligned}\text{Ind}(\widetilde{\gamma}, 0) &= \frac{1}{2\pi i}\int_{\widetilde{\gamma}} \frac{dz}{z} = \frac{1}{2\pi i}\int_a^b \frac{\widetilde{\gamma}'(t)\ dt}{\widetilde{\gamma}(t)} \\ &= \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\gamma'(t)\ dt}{f(\gamma(t))} = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)\ dz}{f(z)}\end{aligned}$$

By Theorem 9.1.1, we are done! ■

There is a reason why this result is called **"argument"** principle. Suppose $\widetilde{\gamma}$ is parametrized by $[0, 1]$. Then we have a continuou function $\theta : [0, 1] \to \mathbb{R}$ that returns an argument of $\widetilde{\gamma}(t)$. Then

$$\text{Ind}(\widetilde{\gamma}, 0) = \frac{\theta(1) - \theta(0)}{2\pi} = \frac{\arg f(\gamma(1)) - \arg f(\gamma(0))}{2\pi}$$

Informally speaking, $\text{Ind}(\widetilde{\gamma}, 0)$ basically measures the change in argument of f as we move around γ. That's why this result is called **"argument"** principle.

Lemma 9.1.3

Suppose $|z - w| < |z| + |w|$ for $w, z \in \mathbb{C}$. Then 0 does not lie

Proof. If we take the triangle with vertices $0, z, w$, by triangle inequality,

$$|z - w| \leq |z - 0| + |w - 0| \implies |z - w| \leq |z| + |w|$$

Here we have the strict inequality. That means the triangle is cannot be a degenerate triangle. So $0, z, w$ are not collinear. ■

Proposition 9.1.4

Suppose γ_0 and γ_1 are two closed curves in $\mathbb{C} \setminus \{0\}$ with the same parameter interval $[a, b]$. If

$$|\gamma_1(t) - \gamma_0(t)| < |\gamma_1(t)| + |\gamma_0(t)|$$

for all $t \in [a, b]$, then $\gamma_0 \simeq_p \gamma_1$. And in particular $\operatorname{Ind}(\gamma_0, 0) = \operatorname{Ind}(\gamma_1, 0)$.

Proof. We shall construct a homotopy $F : [0, 1] \times [a, b] \to \mathbb{C} \setminus \{0\}$ as follows:

$$F(s, t) = s\,\gamma_1(t) + (1 - s)\,\gamma_0(t)$$

For a fixed t, $|\gamma_1(t) - \gamma_0(t)| < |\gamma_1(t)| + |\gamma_0(t)|$ gives us that 0 does not lie in the line segment joining $\gamma_0(t)$ and $\gamma_1(t)$ (Lemma 9.1.3). Therefore, F is a homotopy between γ_0 and γ_1. So $\gamma_0 \simeq_p \gamma_1$.

Furthermore, since γ_0 and γ_1 are homotopic, by Corollary 7.2.4, $\operatorname{Ind}(\gamma_0, 0) = \operatorname{Ind}(\gamma_1, 0)$. ■

Theorem 9.1.5 (Rouche's Theorem)

Let γ be a curve in V such that $\operatorname{Ind}(\gamma, z)$ is either 0 or 1 for $z \in \mathbb{C} \setminus \gamma^*$; and $\operatorname{Ind}(\gamma, z) = 0$ for $z \in C \setminus V$. Let $\Omega = \{z \in V \;:\; \operatorname{Ind}(\gamma, z) = 1\}$. If $f, g \in H(V)$ and

$$|f(z) - g(z)| < |f(z)| + |g(z)|$$

for all $z \in \gamma^*$. Then f and g have the same number of zeroes in Ω.

Proof. Let $\widetilde{\gamma_f} = f \circ \gamma$ and $\widetilde{\gamma_g} = g \circ \gamma$. By the given condition,

$$|\widetilde{\gamma_f}(t) - \widetilde{\gamma_g}(t)| < |\widetilde{\gamma_f}(t)| + |\widetilde{\gamma_g}(t)|$$

Then by Proposition 9.1.4, $\operatorname{Ind}(\widetilde{\gamma_f}, 0) = \operatorname{Ind}(\widetilde{\gamma_g}, 0)$. But the number of zeroes of f in Ω is just $\operatorname{Ind}(\widetilde{\gamma_f}, 0)$ (Argument Principle). Similarly, the number of zeroes of g in Ω is $\operatorname{Ind}(\widetilde{\gamma_g}, 0)$.

Therefore, f and g have the same number of zeroes in Ω. ■

There is another version of Rouche's Theorem that assumes a stronger hypothesis:

$$|f(z) - g(z)| < |f(z)|$$

The proof of this version is left as an exercise for the reader.

§9.2 Open Mapping Theorem

Theorem 9.2.1 (Open Mapping Theorem)

Let $V \subseteq \mathbb{C}$ be a domain and $f \in H(V)$ be non-constant. Then f is an open map (maps open sets to open sets).

Proof. Suppose $S \subseteq V$ is open, we want to show that $f(S)$ is open. Let $z_0 \in S$, and $g(z) = f(z) - f(z_0)$. We choose $r > 0$ such that $\overline{D(z_0, r)} \subseteq S$ and g has no zeroes on $\partial D(z_0, r)$. We can choose such r because we've proved that the zeroes of a holomorphic function are isolated.

Let $\gamma = \partial D(z_0, r)$ and $\delta = \inf\{|g(z)| : z \in \gamma^*\}$. $\delta > 0$ as g has no zeroes on γ^*. Let $h \in \mathbb{C}$ with $|h| < \delta$. Then

$$|(g(z) - h) - g(z)| = |h| < \delta \leq |g(z)|$$

for all $z \in \gamma^*$. By Rouche's Theorem, g and $g - h$ have the same number of zeroes in $D(z_0, r)$. $g(z_0) = 0$, so it has at least one zero in $D(z_0, r)$. Therefore, $g - h$ has at least one zero in $D(z_0, r)$. In other words, there exists $z \in D(z_0, r)$ such that

$$0 = g(z) - h = f(z) - f(z_0) - h \implies g(z) = f(z_0) + h$$

This is true for every $h \in D(0, \delta)$. Therefore,

$$D(f(z_0), \delta) = f(z_0) + D(0, \delta) \subseteq f(D(z_0, r)) \subseteq f(S)$$

For every $f(z_0) \in f(S)$, we can find a ball around $f(z_0)$ that is contained in $f(S)$. So $f(S)$ is open. ■

However, Open Mapping Theorem is not true in real analysis. For example, let's take $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. $(-\alpha, \alpha)$ is an open subset of $\mathbb{R}$. But its image $[0, \alpha^2)$ is not open, despite f being differentiable.

In the beginning of this chapter, we said that Open Mapping Theorem trivializes Problem 2(b) of HW2. Let's see how.

Suppose f is holomorphic in a domain V. If $\operatorname{Re} f$ is constant, then $f(V)$ is a line parallel to the imaginary axis. This is not open, since open sets contain some open disks. V is open, but $f(V)$ is not. So f is not an open map. This contradicts Open Mapping Theorem, so f cannot be non-constant.

Now suppose $\operatorname{Im} f$ is constant. Then $f(V)$ is a line parallel to the real axis, which is not open. V is open, but $f(V)$ is not. So f is not an open map. This contradicts Open Mapping Theorem, so f cannot be non-constant.

If $|f|$ is constant, then $f(V)$ is a circular arc, which is not open. V is open, but $f(V)$ is not. So f is not an open map. This contradicts Open Mapping Theorem, so f cannot be non-constant.

Finally, if $\arg f$ is constant, $f(V)$ is a ray through the origin. This does not contain any open disk, so it's not open. V is open, but $f(V)$ is not. So f is not an open map. This contradicts Open Mapping Theorem, so f cannot be non-constant.

In general, from Open Mapping Theorem, one can conclude that if f is nonconstant holomorphic on V and $D(z_0, r) \subseteq V$, then $f(D(z_0, r))$ cannot be a one-dimensional submanifold[1] embedded in $\mathbb{C}$. In other words, $\dim f(V)$ cannot be 1.

Open Mapping Theorem gives us an elegant proof of Maximum Modulus Principle.

Theorem 9.2.2 (Maximum Modulus Principle)

If f is non-constant holomorphic is a domain V, then $|f|$ cannot attain a maximum in V.

Proof. Assume for the sake of contradiction that $|f(z_0)| \geq |f(V)|$ for all $z \in V$. Since f is non-constant holomorphic, it's an open mapping. Take $r > 0$ with $D(z_0, r) \subseteq V$. Then $f(D(z_0, r))$ is an open set containing $f(z_0)$. So there exists $r' > 0$ such that

$$D(f(z_0), r') \subseteq f(D(z_0, r))$$

[1]You can ignore it if you don't know what it means.

Let $f(z_0) = a + ib$. If $a < 0$, we take $z_1 = f(z_0) - \frac{r'}{2} \in D(f(z_0), r')$. Otherwise, if $a \geq 0$, we take $z_1 = f(z_0) + \frac{r'}{2} \in D(f(z_0), r')$. Then a quick check shows that $|z_1| > |f(z_0)|$. Since $z_1 \in D(f(z_0), r') \subseteq f(D(z_0, r))$, there exists $z_2 \in D(z_0, r)$ such that $f(z_2) = z_1$. Hence

$$|f(z_2)| = |z_1| > |f(z_0)|$$

which contradicts the assumption that $|f(z_0)|$ is maximum. So $|f|$ cannot attain a maximum. ■

Theorem 9.2.3 (Inverse Function Theorem)

Suppose $f : V \to \mathbb{C}$ is injective and holomorphic, and $f'(z) \neq 0$ for every $z \in V$. If $g : f(V) \to V$ is the inverse of f, then g is holomorphic with $g'(z) = \frac{1}{f'(g(z))}$.

Proof. Firstly, let's show that g is continuous. Let $U \subseteq V$ be open. Then by Open Mapping Theorem, $f(U)$ is open. Therefore,

$$g^{-1}(U) = f(U) \quad \text{is open} \quad \implies g \text{ is continuous.}$$

To see that g is holomorphic, let $w_0 \in f(U)$, and $z_0 = g(w_0)$. Since both g and f are continuous, if $w \to w_0$ then $g(w) \to z_0$. If $g(w) = z$, then $f(z) = w$.

$$\lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0} = \lim_{g(w) \to z_0} \frac{g(w) - z_0}{w - f(z_0)} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{f'(z_0)}$$

Therefore, $g'(w_0)$ exists, and it is equal to $\frac{1}{f'(g(w_0))}$. ■

10 Homeworks and Exams

§10.1 Homework 1

Problem 1. Let $\omega = e^{\frac{2\pi i}{3}}$. Suppose $a, b, c \in \mathbb{Z}$. They are nonzero, and not all equal. What is the minimum value of

$$\left|a + b\omega + c\omega^2\right|$$

Problem 2. Show that $SO(2)$ is isomorphic to S^1 where the group operation in $SO(2)$ is the matrix multiplication and the group operation in S^1 is the complex number multiplication.

Problem 3. $t = \cos(\theta) + \mathbf{u}\sin(\theta)$ where $\mathbf{u}$ is a "pure imaginary" unit quaternion. Show that the map

$$\begin{aligned} c_t : H &\to H \\ q &\mapsto t^{-1}qt \end{aligned}$$

rotates $\mathbb{R}^3(\equiv \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k})$ through angle -2θ about the axis $\mathbf{u}$.

Problem 4. Show that

$$\begin{aligned} f : S^1 &\to S^1 \\ z &\mapsto z^n \end{aligned}$$

is a group homomorphism. What's its kernel?

Problem 5. Let $z_0 \in \mathbb{C}$ and $|z_0| < a$ for some positive real number a. Observe that

$$E = \{z \in \mathbb{C} \ : \ |z - z_0| + |z + z_0| = 2a\}$$

represents a very familiar graph. Identify the graph and write a couple of facts about the graph.

§10.2 Homework 2

Problem 1. Find the radius of convergence of the following power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$$

What happens if $z = i$?

Problem 2. Prove that

(a) A holomorphic function f on domain Ω whose derivative vanishes identically is a constant.

(b) The same conclusion holds if

(i) real part is constant.

(ii) imaginary part is constant.

(iii) modulus of f is constant.

(iv) the argument of f is constant.

Problem 3. Let V be open in $\mathbb{C}$ and Γ_1, Γ_2 be two piecewise C^1 paths from $\alpha \in V$ to $\beta \in V$. Suppose the map $f : V \to \mathbb{C}$ has a primitive in V. Show that

$$\int_{\Gamma_1} f(z)\ dz = \int_{\Gamma_2} f(z)\ dz$$

Problem 4. Given $k \in \mathbb{Z}$ and $z, w \in \mathbb{C}$ satisfying $2k\pi < \operatorname{Im}(z) \leq 2(k+1)\pi$ and $2k\pi < \operatorname{Im}(w) \leq 2(k+1)\pi$. Then $e^z = e^w$ if and only if $z = w$.

Problem 5. Define $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ as the upper half plane in $\mathbb{C}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ as the unit disk of $\mathbb{C}$. Consider, the Cayley map, $C(z) = \frac{z-i}{z+i}$. Provide explicit description of the following sets:

(a) $C(\mathbb{R})$

(b) $C(\mathbb{H})$

(c) $C^{-1}(\mathbb{D})$

Problem 6. Prove the following:

(i) $f(0) = 0, f(z) = e^{-z^{-4}}$, when $z \neq 0$. Prove that the partial derivatives exist at every point (including the origin) and f satisfies the CR-equations at every point. Then prove that f is not complex differentiable at the origin.

(ii) Prove that $f(z) = ix^2 - 2xy - iy^2 + 3x + 3iy + i$ can be written as a function of z, and show that f is holomorphic. $(z = x + iy)$

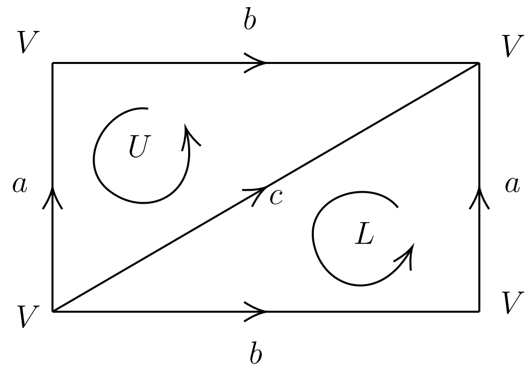
(iii) Consider the function $f(z) = z^m(\bar{z})^n$ for non-negative integers m and n. Is f holomorphic in any open subset of $\mathbb{C}$?

Problem 7. Compute $\int_\gamma \frac{dz}{z-z_0}$, where $\gamma(t) = z_0 + re^{it}, 0 \leq t \leq 2\pi$ and $r > 0$. Using this conclude that $\frac{1}{z-z_0}$ doesn't have a primitive in $\{z : |z - z_0| < r\}$.

Problem 8. Suppose $V \subset \mathbb{C}$ be open and convex, and $f : V \to \mathbb{C}$ is a continuous function with $\int_{\partial T} f(z)dz = 0$ for every triangle $T \subset V$. Then show that there exists a function $g : V \to \mathbb{C}$ such that $g' = f$ in V.

Problem 9. Compute the simplicial homology groups of the triangular parachute obtained from Δ^2 (standard 2-simplex) by identifying its three vertices to a single point.

Problem 10. Consider the following Δ-complex structure of T^2 (torus):



Compute the cellular homology groups of T^2. Follow the exact procedure that we did for simplicial homology groups computation.

§10.3 Midterm 1 / Homework 3

Problem 1. Prove the following:

(a) Let Ω be a domain in $\mathbb{C}$, and $f : \Omega \to \mathbb{C}$ be a C^1 function such that

$$i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$$

Is f holomorphic?

(b) Consider $V = \{z \in \mathbb{C} \ : \ \operatorname{Re} z > 1\}$. Show that

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

is holomorphic on V.

Problem 2. Show that $\sum\limits_{n=0}^{\infty} n^n z^{n^n}$ has radius of convergence 1, but $\sum\limits_{n=0}^{\infty} n^n z^n$ is not convergent at all.

Problem 3. Prove the following:

(a) Let $V \subseteq C$ be a domain, and $f : V \to \mathbb{C}$ be a continuous function with the property

$$\int_\gamma f(z)\, dz = 0$$

for any closed contour γ in V. Show that f has an antiderivative on V.

(b) Does $f(z) = \frac{1}{z}$ have antiderivative on $\mathbb{C} \setminus \{0\}$?

Problem 4. Let z_0, z_0' be two nonzero complex numbers. A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is defined in a way so that

$$f(z) = f(z + mz_0 + nz_0')$$

for any $z \in \mathbb{C}$, for all $m, n \in \mathbb{Z}$. Show that f is constant.

Problem 5. Let $n \in \mathbb{N}$. Find all entire functions f for which there exist $M, R > 0$ such that $|f(z)| \geq M |z|^n$ whenever $|z| > R$.

Problem 5 (Alternate)**.** Let f be an entire function with the property that $\left|f\left(\frac{1}{n}\right)\right| \leq n^{-n}$ for $n \in \mathbb{N}$. Show that f is constant.

Problem 6. Let f be an entire function such that $f(ix) = f(x)$ for all $x \in (1, 2) \subseteq \mathbb{R}$. Prove that

$$f(z) = f(-z) \qquad \forall\, z \in \mathbb{C}$$

Problem 6 (Alternate)**.** Let U be a non-empty open connected subset of $\mathbb{C}$. Let $p \in U$ and $\Delta_r(p)$ be the disk of radius r (> 0) centered at p. $\overline{\Delta_r(p)} \subseteq U$, and $f : U \to \mathbb{C}$ be holomorphic. Prove that the average of f on $\Delta_r(p)$ is $f(p)$. In other words,

$$\frac{1}{\pi r^2} \int_{\Delta_r(p)} f(z)\; dA = f(p)$$

[Hint: Convert the integral suitably into a double integral and use the "mean-value property". Recall $dA = dx\, dy = r\, dr\, d\theta$, where $z = re^{i\theta} = x + iy$.]

Problem 7. Let V be a star-convexdomain, and $f : V \to \mathbb{C}$ be a holomorphic function. Prove that

$$\int_\gamma f(z)\; dz = 0$$

for any closed contour γ in V.

Problem 8. Let $V \subseteq C$ be open in $\mathbb{C}$, and γ_1, γ_2 be two contours in V that are homotopic with fixed end points. Then show that

$$\int_{\gamma_1} f(z)\; dz = \int_{\gamma_2} f(z)\; dz$$

for any holomorphic function $f : V \to \mathbb{C}$.

Problem 9. Prove that

$$\int_\gamma P(z)\; dz = 0$$

for any polynomial P and any chain γ with the property: $\partial(\gamma) = 0$ where $\partial : C_1 \to C_0$ is the usual boundary map.

Problem 10. Compute the following integral for the following contours:

$$\int_\gamma \frac{\cos(z^2) + z}{z - \sqrt{\pi}}\; dz$$

(a) $\gamma(t) = 2e^{it}$, for $0 \le t \le 2\pi$

(b) $\gamma(t) = e^{2\pi i t}$, for $0 \le t \le 1$

(c) $\gamma(t) = (1 + i) + 5e^{2\pi i t}$, for $0 \le t \le 1$

§10.4 Midterm 2 / Homework 4

Notations:

1. $f \in H(V)$ means f is holomorphic on V.
2. $f \in C(V)$ means f is continuous on V.
3. $f \in C\left(\overline{V}\right) \cap H(V)$ means $f \in C\left(\overline{V}\right)$ and $f\big|_V \in H(V)$.
4. $D'(z, r) = D(z, r) \setminus \{z\} = \{w \in \mathbb{C} \;:\; 0 < |w - z| < r\}$

Problem 1. Suppose that $f \in H(\mathbb{C})$ and $|f(z)| \le e^{\operatorname{Re} z}$ for all z. Show that $f(z) = ce^z$ for some constant c.

Problem 2. Suppose that $f \in H(\mathbb{C})$. n is a positive integer and $|f(z)| \le (1+|z|)^n$ for all z. Show that f is a polynomial.

Problem 3. Suppose that $f, g \in H(D(z,r))$, $1 \le m \le n$, f has a zero of order n at z and g has a zero of order m at z. Show that f/g has a removable singularity at z.

Problem 4. Suppose that $f \in H(\mathbb{C})$ and $f(n) = 0$ for all $n \in \mathbb{Z}$. Show that all the singularities of $f(z)/\sin(\pi z)$ are removable.

Problem 5. Suppose that $f \in H(\mathbb{C})$, $f(z+1) = -f(z)$ for all z, $f(0) = 0$, and $|f(z)| \le e^{\pi|\operatorname{Im} z|}$ for all z. Show that $f(z) = c\sin(\pi z)$ for some constant c.

Problem 6. Let $V = \mathbb{C} \setminus \{0\}$. Show that there does not exist $f \in H(V)$ such that $e^{f(z)} = z$ for all $z \in V$.

Problem 7. Suppose that V is a bounded open subset of the plane and $f \in C\left(\overline{V}\right) \cap H(V)$. Show that if $M \ge 0$ and $|f(z)| \le M$ for all $z \in \partial V$, then $|f(z)| \le M$ for all $z \in V$.

Problem 8. Let $V = D(0,1)$. Suppose that $f \in C\left(\overline{V}\right) \cap H(V)$, $f(0) = 0$ and $|f(z)| \le 1$ for all $z \in \overline{V}$. Show that $|f(z)| \le |z|$ for all $z \in V$.

Problem 9. Let $V = D(0,1)$. Suppose that $f \in H(V)$, $f(0) = 0$ and $|f(z)| \le 1$ for all $z \in V$. Show that $|f(z)| \le |z|$ for all $z \in V$.

Problem 10. Suppose that $\alpha < 1$ and $f \in H\left(D'\left(z_0, r\right)\right)$ satisfies

$$|f(z)| \le c\,|z - z_0|^{-\alpha}\,.$$

Show that f has a removable singularity at z_0.

§10.5 Homework 5

Problem 1. Let $a, b, c \in \mathbb{C}$ be three non-collinear complex numbers. Show that they form an equilateral triangle iff $a^2 + b^2 + c^2 = ab + bc + ca$.

Problem 2. Let $a, b \in \mathbb{C}$ be two complex numbers with the property: $\operatorname{Re} a \le 0$ and $\operatorname{Re} b \le 0$. Show that $\left|e^a - e^b\right| \le |a - b|$.

Problem 3. Consider the following power series $\sum\limits_{n=0}^{\infty} a_n z^n$. The coefficients a_n are real, and given by the following recurrence relation:

$$3a_n + 4a_{n-1} - a_{n-2} = 0\ , \qquad \forall\, n \ge 2$$

$a_0 = 1$ and $a_1 = -1$. Find the radius of convergence of this power series and the function to which this series converges.

Problem 4. Show that

$$f''(z) = zf(z)\ , \qquad f'(0) = f(0) = 1$$

has a unique entire solution.

Problem 5. Find a conformal map[1] f such that

$$f(\{z : |z| < 1 \text{ and } \operatorname{Re} z > 0\}) = \{z : |z| < 1\}.$$

Problem 6. Find the area of the image of $D(0,1)$ under tha map $f(z) = z + \frac{z^2}{2}$.

Problem 7. Suppose f is holomorphic on $D(0,1)$, $f(0) = 3 + 4i$, and $|f(z)| \le 5$ for every $z \in D(0,1)$. Find $f'(0)$.

Problem 8. Suppose $f \in H(\mathbb{C})$ such that

$$\int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right| \, d\theta \le r^{\frac{17}{3}} , \quad \text{for every } r > 0.$$

Show that $f \equiv 0$.

Problem 9. Let $f \in H(\mathbb{C})$ with the property

$$|f(z)| \le |\operatorname{Re} z|^{-\frac{1}{2}} \quad \text{off the imaginary axis.}$$

Prove that f is constant.

Problem 10. Suppose $f \in H(\mathbb{C})$, $f(0) = 0$, and

$$\{z : |f(z)| < M\} \quad \text{is connected} \quad \forall\, M > 0.$$

Show that $f(z) = cz^n$ for some $c \in \mathbb{C}$ and $n \in \mathbb{N}$.

§10.6 Homework 6

Problem 1. Suppose f is holomorphic on $\mathbb{D} = D(0,1)$, and $f : \mathbb{D} \to \mathbb{D}$. Show that

$$|f'(z)| \le \frac{1}{1 - |z|^2}.$$

Problem 2. Suppose $f, h \in H(\mathbb{C})$, and for every $z \in \mathbb{C}$, we have

$$|f(z)| \le |g(z)|.$$

Show that $f(z) = c\, g(z)$ for some constant $c \in \mathbb{C}$.

Problem 3. If $f, h \in H(\mathbb{C})$, and

$$\operatorname{Re} f \le k \;\operatorname{Re} g \quad \text{for some } k,$$

then show that there exists $a, b \in \mathbb{C}$ such that $f(z) = a\, g(z) + b$.

Problem 4. Suppose f is complex valued continuous function on $[0,1]$, and

$$g(z) = \int_0^1 f(t)\, e^{tz} \, dt , \quad \forall\, z \in \mathbb{C}.$$

Show that $g \in H(\mathbb{C})$.

[1] A **conformal map** is a function that preserves angle.

Problem 5. Let $f \in C^0(\mathbb{C})$ be holomorphic on $\{z : \operatorname{Re} z \neq 0\}$. Show that f is entire.

Problem 6. Suppose $u_{xx} + u_{yy} = 0$ for some positive function u on $\mathbb{R}^2$. Show that u is constant.

Problem 7. Show that $P(z) = z^{47} - z^{23} + 2z^{11} - z^5 + 4z^2 + 1$ has at least one zero in $\mathbb{D}$.

Problem 8. How many zeroes does $f(z) = 3z^{100} - e^z$ have in $\mathbb{D}$? Are all distinct?

Problem 9. How many zeroes does $f(z) = 3z^9 + 8z^6 + z^5 + 2z^3 + 1$ have in $\{z : 1 < |z| < 2\}$?

Problem 10. If $f \in H(\mathbb{C})$, then show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Problem 11. Find all entire functions such that $f(z^k) = (f(z))^k$ for $k > 1$.

Problem 12. Find all holomorphic functions on $\mathbb{D}$ that satisfy

$$f''\left(\frac{1}{n}\right) + f\left(\frac{1}{n}\right) = 0 \quad \forall n \geq 2.$$

Problem 13. Let f be a holomorphic function, and

$$f(z) = f\left(\frac{1}{z}\right) \quad \forall z \neq 0.$$

Suppose f is real on $\partial\mathbb{D}$. Show that $f(z)$ is real for all nonzero real z.

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